Symmetry and Quantum Information February 19, 2018

Problem Set 2

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Problem 1 (Symmetries of ebit and singlet, 3 points).
Let $|\Phi^+_{AB}\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ denote the *ebit* state and $|\Psi^-_{AB}\rangle := \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ the *singlet* state.

(a) Show that the ebit state has the following symmetry: $(X \otimes I)|\Phi^+_{AB}\rangle = (I \otimes X^T)|\Phi^+_{AB}\rangle$ for every operator X.

(b) Using part (a), deduce that $(U \otimes \bar{U})|\Phi^+_{AB}\rangle = |\Phi^+_{AB}\rangle$ for every unitary U.

(c) Show that the singlet state has the following symmetry: $(X \otimes X)|\Psi^-_{AB}\rangle = \det(X)|\Psi^-_{AB}\rangle$ for every operator X.

Problem 2 (Product states yield independent measurement outcomes, 3 points).
Suppose that Alice and Bob share a quantum state $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice performs a projective measurement $\{Q_{A,x}\}$ on her system and Bob a projective measurement $\{R_{B,y}\}$ on his system. The order of measurement is not important, since they measure on separate subsystems.

(a) Verify that the joint probability that Alice' measurement outcome is x and Bob's measurement outcome is y is given by

$$p(x,y) = \langle \Psi_{AB}|Q_{A,x} \otimes R_{B,y}|\Psi_{AB}\rangle. \tag{2.1}$$

(b) Now assume that $|\Psi_{AB}\rangle$ is a product state, i.e., $|\Psi_{AB}\rangle = |\psi_A\rangle \otimes |\phi_B\rangle$. Using formula (2.1), conclude that in this case the measurement outcomes of Alice and Bob are independent.

Hint: Recall that two random variables are called independent if their joint probability distribution is of the form $p(x,y) = q(x)r(y)$.

Problem 3 (Classical and quantum strategies for the GHZ game, 6 points).
Three players and the referee play the GHZ game, following the same conventions as in Lecture 3. In particular, the referee chooses each of the four questions xyz with equal probability 1/4.

(a) Verify that the winning probability for a general quantum strategy, specified in terms of a state $|\psi\rangle_{ABC}$ and observables A_x, B_y, C_z, is given by

$$p_{\text{win,q}} = \frac{1}{2} + \frac{1}{8}\langle \psi_{ABC}|A_0 \otimes B_0 \otimes C_0 - A_1 \otimes B_1 \otimes C_0 - A_1 \otimes B_0 \otimes C_1 - A_0 \otimes B_1 \otimes C_1|\psi_{ABC}\rangle. \tag{2.2}$$

(b) Suppose that Alice, Bob, and Charlie play the following randomized classical strategy: When they meet before the game is started, they flip a biased coin. Let π denote the probability that the coin comes up heads. Depending on the outcome of the coin flip, which we denote by $\lambda \in \{\text{HEADS}, \text{TAILS}\}$, they use one of two possible deterministic strategies $a_\lambda(x), b_\lambda(y), c_\lambda(z)$ to play the game. Find a formula analogous to (2.2) for the winning probability $p_{\text{win,cl}}$ of their strategy.

(c) In class we discussed that even randomized classical strategies such as described in (b) cannot do better than $p_{\text{win,cl}} \leq 3/4$. Verify this explicitly using the formula you derived in (b).

(d) Any classical strategy can be realized by a quantum strategy. Show this explicitly for the randomized classical strategy described in (b) by constructing a quantum state $|\psi\rangle_{ABC}$ and observables A_x, B_y, C_z such that $p_{\text{win,cl}} = p_{\text{win,q}}$.