**Problem 1** (Pure state entanglement, 3 points).
In class we observed that a pure state $|\Psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is unentangled if and only if its reduced density operators $\rho_A$ and $\rho_B$ are pure states. Here you will generalize this observation and show that the maximal fidelity squared between $|\Psi_{AB}\rangle$ and any product state is given by the largest eigenvalue of $\rho_A$, denoted $\lambda_{\max}(\rho_A)$. That is, show that
\[
\max_{|\phi_A\rangle = |\psi_B\rangle} |\langle \Psi_{AB} | \phi_A \otimes \psi_B \rangle|^2 = \lambda_{\max}(\rho_A).
\]
*Hint: Use the Schmidt decomposition discussed in Lecture 8.*

**Problem 2** (De Finetti and mean field theory, 4 points).
In this exercise you will explore the consequences of the quantum de Finetti theorem for mean field theory. Consider a Hermitian operator $h$ on $\mathbb{C}^d \otimes \mathbb{C}^d$ and the corresponding mean-field Hamiltonian, i.e.,
\[
H = \frac{1}{n-1} \sum_{i\neq j} h_{i,j}
\]
on $(\mathbb{C}^d)^\otimes n$, where each term $h_{i,j}$ acts by the operator $h$ on subsystems $i$ and $j$ and by the identity operator on the remaining subsystems (e.g., $h_{1,2} = h \otimes I^\otimes (n-2)$).

(a) Show that the eigenspaces of $H$ are invariant subspaces for the action of the symmetric group.

Now assume that the eigenspace with minimal eigenvalue (the so-called *ground space*) is nondegenerate and spanned by some $|E_0\rangle$, with corresponding eigenvalue $E_0$. Then part (a) implies that $R_\pi |E_0\rangle = \chi(\pi) |E_0\rangle$ for some function $\chi$. This function necessarily satisfies $\chi(\pi \tau \pi^{-1}) = \chi(\pi) \chi(\tau)$.

(b) Show that $\chi(i \leftrightarrow j) = \chi(1 \leftrightarrow 2)$ for all $i \neq j$. Conclude that $|E_0\rangle$ is either a symmetric tensor or an antisymmetric tensor.

*Hint: First show that $\chi(\pi \tau \pi^{-1}) = \chi(\tau).*

If $n > d$, then there exist no nonzero antisymmetric tensors. Thus, in the so-called *thermodynamic limit* of large $n$, the ground state $|E_0\rangle$ is in the symmetric subspace $\text{Sym}^n(\mathbb{C}^d)$ and so the quantum de Finetti theorem is applicable.

(c) Show that, for large $n$, the energy density in the ground state can be well approximated by minimizing over tensor power states. That is, show that
\[
\frac{E_0}{n} \approx \min_{|\psi\rangle} \langle \psi^{\otimes 2} | h | \psi^{\otimes 2} \rangle = \frac{1}{n} \min_{|\psi\rangle} \langle \psi^{\otimes n} | H | \psi^{\otimes n} \rangle.
\]

*Hint: The following fact about the trace distance will be useful. If $\rho, \sigma$ are density operators and $O$ an observable, then $|\text{tr}[O\rho] - \text{tr}[O\sigma]| \leq 2 |O|_\infty T(\rho, \sigma)$, where $|O|_\infty := \max_{|\phi\rangle = 1} |\langle \phi | O | \phi \rangle|$. This justifies the folklore that “in the mean field limit the ground state has the form $|\psi\rangle^{\otimes \infty}$.”
Problem 3 (The antisymmetric state, 5 points).
In class, we discussed the quantum de Finetti theorem for the symmetric subspace. It asserts that the reduced density operators $\rho_{A_1\ldots A_k}$ of a state on $\Sym^k\otimes(C^D)$ are $\sqrt{kD/n}$ close in trace distance to a separable state (in fact, to a mixture of tensor power states).

The goal of this exercise is to show that some kind of dependence on the dimension $D$ is unavoidable in the statement of the theorem. To start, consider the Slater determinant

$$|S\rangle_{A_1\ldots A_d} = |1\rangle \wedge \cdots \wedge |d\rangle := \sqrt{1/d!} \sum_{\pi \in S_d} \text{sign}(\pi) |\pi(1)\rangle \otimes \cdots \otimes |\pi(d)\rangle \in (C^d)^{\otimes d}.$$ 

We define the antisymmetric state on $C^d \otimes C^d$ by tracing out all but two subsystems,

$$\rho_{A_1A_2} = \text{tr}_{A_3\ldots A_d} [ |S\rangle \langle S |].$$

(a) Let $F = R_{1\leftrightarrow 2}$ denote the swap operator on $(C^d)^{\otimes 2}$. Prove the following identity, which is known as the swap trick:

$$\text{tr}[F(\sigma \otimes \gamma)] = \text{tr}[\sigma \gamma]$$

(b) Show that $T(\rho_{A_1A_2}, \sigma_{A_1A_2}) \geq \frac{1}{2}$ for all separable states $\sigma_{A_1A_2}$.

Hint: Consider the POVM element $Q = \Pi_2$ (i.e., the projector onto the symmetric subspace).

Thus you have shown that the antisymmetric state is far from any separable state. However, note that $|S\rangle$ is not in the symmetric subspace.

(c) Show that $|S\rangle^{\otimes 2} \in \Sym^d(C^d \otimes C^d)$, while $\rho_{A_1A_2}^{\otimes 2}$ is likewise far away from any separable state. Conclude that the quantum de Finetti theorem must have some dimension dependence.

Hint: $|S\rangle^{\otimes 2}$ is a state of $2d$ quantum systems that we might label $A_1\ldots A_dA'_1\ldots A'_d$ (the unprimed systems refer to the first copy of $|S\rangle$ and the primed to the second). Let the permutation group $S_d$ act by simultaneously permuting unprimed and primed systems and show that $|S\rangle^{\otimes 2}$ is in the corresponding symmetric subspace. Similarly, $\rho^{\otimes 2}$ is an operator on $A_1A_2A'_1A'_2$. How do you need to partition the systems so that $\rho^{\otimes 2}$ is far from being separable?

Problem 4 (Classical data compression, 4 points).
In this exercise you will show that the Shannon entropy $h(p) = -p \log p - (1-p) \log(1-p)$ is the optimal compression rate for the coin flip problem discussed in class. Assume that Alice compresses her random sequence of $n$ coin flips by applying a function $E_n: \{H, T\}^n \rightarrow \{0, 1\}^{[nR]}$, and Bob decompresses by applying a corresponding function $D_n: \{0, 1\}^{[nR]} \rightarrow \{H, T\}^n$.

(a) Which are the coin flip sequences that are transmitted correctly? Find an upper bound on their cardinality in terms of $R$.

(b) Show that, if $R < h(p)$, then the probability of success tends to zero for large $n$.

Hint: Distinguish between typical and atypical sequences of coin flips.
The following exercises are offered as additional opportunity for practice. They will not be graded.

Optional Problem 5 (Entanglement witness for the ebit).
Recall that an entanglement witness for a quantum state $\rho_{AB}$ is an observable $O_{AB}$ such that $\text{tr}[O_{AB}\rho_{AB}] > 0$, while $\text{tr}[O_{AB}\sigma_{AB}] \leq 0$ for every separable state $\sigma_{AB}$. Construct an entanglement witness for the ebit state $|\Phi_{AB}^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

Hint: Use the claim of Problem 7 to your advantage!

Optional Problem 6 (Trace distance and observables). In this problem, you will show that density operators $\rho$ and $\sigma$ with small trace distance $T(\rho, \sigma)$ have similar expectation values.

(a) Show that, for every two Hermitian operators $M$ and $N$, $|\text{tr}[MN]| \leq ||M||_1||N||_\infty$. Here, $||M||_1$ is the trace norm that you know from class (i.e., the sum of absolute values of the eigenvalues of $M$) and $||N||_\infty := \max_{|\phi|=1} ||\phi|N|\phi||$ is the operator norm (which can also be defined as the maximal absolute value of the eigenvalues of $N$).

(b) Conclude that, for every observable $O$, $|\text{tr}[\rho O] - \text{tr}[\sigma O]| \leq 2 ||O||_\infty T(\rho, \sigma)$.

This confirms the hint given in Problem 2 part (c).

(c) Find a (nonzero) observable for which the bound in part (b) is an equality.