Quantum Information Theory, Spring 2019

Exercise Set 1 **in-class practice problems**

1. **Dirac notation quiz:** In the Dirac notation, every vector is denoted by a ket (i.e., $|\cdot\rangle$) and every linear functional is denoted by a bra (i.e., $\langle\cdot|$). One can think of kets as column vectors and bras as row vectors. Hence, if $|\psi\rangle$ is a column vector, then $\langle\psi|$ denotes the row vector obtained by taking the conjugate transpose of the column vector.

   (a) Let $|\psi\rangle$ and $|\phi\rangle$ be vectors in $\mathbb{C}^n$ and A an $n \times n$ matrix. Which of the following expressions are syntactically correct? For those that do, what kind of object do they represent (e.g., numbers, vectors, ...)? Can you write them using 'ordinary' notation?

i. $\|\psi\rangle + \langle\phi\|$	iv. $\langle\psi\| A$	vii. $\|\psi\rangle \langle\phi\| A$	x. $\langle\psi\| A \|\phi\rangle + \langle\psi\|\phi\rangle$
ii. $\|\psi\rangle \langle\phi\|$	v. $\langle\psi\| A + \langle\psi\|$	viii. $\|\psi\rangle A \langle\phi\|$	xi. $\langle\psi\|\phi\rangle \langle\psi\|$
iii. $A \langle\psi\|$	vi. $\|\psi\rangle \langle\phi\| + A$	ix. $\langle\psi\| A \|\phi\rangle$	xii. $\langle\psi\|\phi\rangle A$

   (b) Let $\rho = |\psi\rangle \langle\psi|$ and $\sigma = |\phi\rangle \langle\phi|$ be two pure states on the same system. Verify that

   $$\operatorname{tr}[\rho\sigma] = |\langle\psi|\phi\rangle|^2.$$

   Hint: You may use that the trace is cyclic, i.e. $\operatorname{tr}[ABC] = \operatorname{tr}[CAB] = \operatorname{tr}[BCA]$.

Solution.

(a)
   i. Not syntactically correct.
   ii. Syntactically correct. The resulting object is an $n \times n$-matrix: $\psi\phi^*$.
   iii. Not syntactically correct.
   iv. Syntactically correct. The resulting object is an n-dimensional row vector: $\phi^* A$.
   v. Syntactically correct. The resulting object is an n-dimensional row vector: $\phi^* A + \psi^*$.
   vi. Syntactically correct. The resulting object is an $n \times n$-matrix: $\psi\phi^* + A$.
   vii. Syntactically correct. The resulting object is an $n \times n$-matrix: $\psi\phi^* A$.
   viii. Not syntactically correct.
   ix. Syntactically correct. The resulting object is a complex number: $\psi^* A\phi$.
   x. Syntactically correct. The resulting object is a complex number: $\psi^* A\phi + \psi^*\phi$.
   xi. Syntactically correct. The resulting object is an n-dimensional row vector: $\psi^*\phi\psi^*$.
   xii. Syntactically correct. The resulting object is an $n \times n$-matrix: $(\psi^*\phi)A$.
   Note that this one is slightly weird, as putting parentheses would yield an undefined expression: $\langle\psi| (|\phi\rangle A)$.

(b) This follows directly from cyclicity of the trace:

$$\operatorname{tr}[\rho\sigma] = \operatorname{tr}[(|\psi\rangle \langle\psi|)(|\phi\rangle \langle\phi|)] = \operatorname{tr}[|\psi\rangle \langle\psi|\phi\rangle \langle\phi|] = \operatorname{tr}[\langle\psi|\phi\rangle \langle\phi|\psi\rangle] = \langle\psi|\phi\rangle \langle\phi|\psi\rangle = |\langle\psi|\phi\rangle|^2$$

□

2. **Bloch sphere bonanza:** Recall from the lecture that one can visualize any one-qubit state ρ by its Bloch vector $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$.

   (a) Let σ be another qubit state, with Bloch vector $\vec{s}$. Verify that

   $$\mathrm{tr}[\rho\sigma] = \frac{1}{2}\left(1 + \vec{r} \cdot \vec{s}\right).$$

   (b) Let $\{|\psi_x\rangle\}_{x=0,1}$ denote an orthonormal basis of $\mathbb{C}^2$, $\mu\colon \{0,1\} \to \mathrm{Pos}(\mathbb{C}^2)$ the corresponding basis measurement (i.e., $\mu(x) = |\psi_x\rangle\langle\psi_x|$ for $x \in \{0,1\}$), and $\vec{r}_x$ the Bloch vector of $|\psi_x\rangle\langle\psi_x|$ for $x \in \{0,1\}$. Show that $\vec{r}_0 = -\vec{r}_1$. Moreover, show that the probability of obtaining outcome $x \in \{0,1\}$ when measuring ρ using μ is given by $\frac{1}{2}(1 + \vec{r} \cdot \vec{r}_x)$. How can you visualize these two facts on the Bloch sphere?

   (c) Now imagine that ρ is an unknown qubit state ρ whose Bloch vector $\vec{r}$ you would like to characterize completely. Consider the following measurement with six outcomes:

   $$\mu\colon \{x,y,z\} \times \{0,1\} \to \mathrm{Pos}(\mathbb{C}^2), \quad \mu(a,b) = \frac{I + (-1)^b \sigma_a}{6},$$

   where $\sigma_x = X$, $\sigma_y = Y$, and $\sigma_z = Z$ are the three Pauli matrices. Show that μ is a valid measurement and that the probabilities of measurement outcomes are given by

   $$p(a,b) = \frac{1 + (-1)^b r_a}{6}.$$

   How can you visualize this formula on the Bloch sphere? Describe how measuring many copies of ρ by using μ allows for estimating the entries of $\vec{r}$ to arbitrary accuracy.

Solution.

(a) We calculate:

$$\begin{aligned}\mathrm{tr}[\rho\sigma] &= \mathrm{tr}\left[\frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)\frac{1}{2}\left(I + s_x X + s_y Y + s_z Z\right)\right] \\ &= \frac{1}{2}\left(1 + r_x s_x + r_y s_y + r_z s_z\right) = \frac{1}{2}\left(1 + \vec{r} \cdot \vec{s}\right).\end{aligned}$$

Here we used that $X^2 = Y^2 = Z^2 = I$ have trace 2, while $XY = iZ$, $YZ = iX$, $ZX = iY$ are traceless and hence do not contribute.

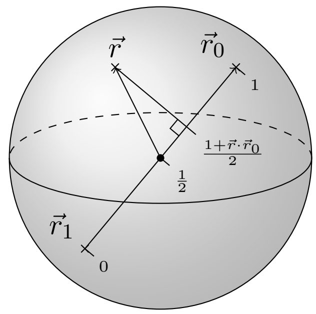
(b) Using 2 (a), then 1 (b), and finally that $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthogonal:

$$\frac{1}{2}\left(1 + \vec{r}_0 \cdot \vec{r}_1\right) = \mathrm{tr}[|\psi_0\rangle\langle\psi_0|\,|\psi_1\rangle\langle\psi_1|] = |\langle\psi_0|\psi_1\rangle|^2 = 0.$$

Thus, $\vec{r}_0 \cdot \vec{r}_1 = -1$. Since $\vec{r}_0$ and $\vec{r}_1$ are unit vectors, this implies that $\vec{r}_0 = -\vec{r}_1$.
The second claim follows directly by using 2 (a) and the rule for the probability of measurement outcomes:

$$p(x) = \mathrm{tr}[\mu(x)\rho] = \frac{1}{2}\left(I + \vec{r}_x \cdot \vec{r}\right)$$

In the Bloch sphere picture, one can associate to this basis measurement a line that goes through the center of the Bloch ball, and punctures the sphere at the tips of the vectors $\vec{r}_0$ and $\vec{r}_1$. One can think of this line as a ruler, having the 0 tick at $\vec{r}_1$ and the 1 tick at $\vec{r}_0$. The probability of measuring 0 can then be found by projecting $\vec{r}$ onto the line, and subsequently writing down at what tick the projected vector ends up.



(c) Each Pauli matrix is Hermitian and squares to the identity matrix, hence has eigenvalues in ± 1. Thus, $\mu(a,b)$ is Hermitian with eigenvalues in $\{0, 1/3\}$, hence positive semidefinite. Since moreover

$$\sum_{a=x,y,z} \sum_{b=0,1} \mu(a,b) = \sum_{a=x,y,z} \frac{I}{3} = I,$$

the map μ defines a measurement.
Next, we calculate:

$$p(a,b) = \operatorname{tr}[\mu(a,b)\rho] = \frac{1}{6}\left(1 + (-1)^b \operatorname{tr}\left[\sigma_a \frac{1}{2}\left(I + r_x X + r_y Y + r_z Z\right)\right]\right) = \frac{1}{6}\left(1 + (-1)^b r_a\right)$$

by a similar calculation as in the proof of part (a). (This can also be seen by noting that $\mu(a,0)$ and $\mu(a,1)$ are proportional by a factor $\frac{1}{3}$ to the operators corresponding to a measurement in the a-axis.)
How can we use this to determine the unknown quantum state? We obtain outcome $(a,0)$ with probability $(1+r_a)/6$ for each $a \in \{x,y,z\}$. If we apply the measurement μ to $N \to \infty$ independent copies of ρ, then

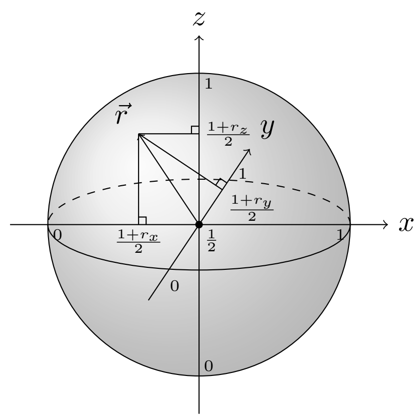
$$\frac{\#(a,0)}{N} \to \frac{1+r_a}{6},$$

in probability by the law of large numbers, where $\#(a,0)$ denotes the number of times that the measurement outcome was $(a,0)$. Thus, we can estimate the components of the Bloch vector of the unknown quantum state by

$$\hat{r}_a = 6\frac{\#(a,0)}{N} - 1.$$

A downside of this protocol is that with some probability our estimate $\vec{r}$ may have norm larger than one, i.e., may not correspond to a quantum state.

This protocol can be viewed on the Bloch sphere as follows. Suppose that one first flips a three-sided coin to uniformly sample one of the three coordinate axes. Then, one performs a basis measurement with respect to the two unit vectors that lie on this coordinate axis. As we have seen in the previous question, the probability distribution of the resulting measurement is then related to the projection of $\vec{r}$ onto this coordinate axis. So, the outcome of the measurement tells us something about the corresponding entry of the vector $\vec{r}$. One can see this in the picture below.



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