Quantum Information Theory, Spring 2019

Exercise Set 10 in-class practice problems

1. Separable maps:
   
   (a) Let $\Xi \in \text{CP}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z} \otimes \mathcal{W})$. Show that $\Xi \in \text{SepCP}(\mathcal{X} : \mathcal{Y}, \mathcal{Z} : \mathcal{W})$ if and only if there exist $A_a \in \text{L}(\mathcal{X}, \mathcal{Z})$ and $B_a \in \text{L}(\mathcal{Y}, \mathcal{W})$ such that
   
   $$\Xi(X) = \sum_{a \in \Sigma} (A_a \otimes B_a)X(A_a \otimes B_a)^*,$$
   
   for all $X \in \text{L}(\mathcal{X} \otimes \mathcal{Y})$.
   
   (b) Let $\Xi_1 \in \text{SepCP}(\mathcal{X}, \mathcal{U} : \mathcal{V}, \mathcal{W})$ and $\Xi_2 \in \text{SepCP}(\mathcal{U}, \mathcal{Z} : \mathcal{V}, \mathcal{W})$. Show that their composition is also separable:
   
   $$\Xi_2 \circ \Xi_1 \in \text{SepCP}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W}).$$

2. Examples of separable maps: Show that the following maps jointly implemented by Alice and Bob are separable:
   
   (a) Alice and Bob share a random variable with probability distribution $p = (p_{a,b} : a \in \Sigma, b \in \Gamma)$, where Alice has the first register and Bob has the second one. Moreover, Alice has a quantum register $A$ and Bob has a quantum register $B$. They both observe their halves of the random variable. If Alice’s value is $a \in \Sigma$, she applies a channel $\Phi_a$ on her register $A$. Similarly, if Bob’s value is $b \in \Gamma$, he applies a channel $\Psi_b$ on his register $B$.
   
   (b) Alice has a register $A$ that she measures in the standard basis. She sends the measurement outcome $a \in \Sigma$ to Bob who applies a channel $\Psi_a$ on his register $B$.
   
   (c) Any LOCC channel.

3. Instruments: Recall that an instrument is a collection $\{\Phi_a : a \in \Sigma\} \subset \text{CP}(\mathcal{X}, \mathcal{Y})$ such that $\sum_{a \in \Sigma} \Phi_a \in \text{C}(\mathcal{X}, \mathcal{Y})$. When applied to state $\rho \in \text{D}(\mathcal{X})$, it produces outcome $a \in \Sigma$ with probability $\text{Tr}[\Phi_a(\rho)]$ and changes the state to $\rho_a = \Phi_a(\rho)/\text{Tr}[\Phi_a(\rho)]$. Show that any instrument can be implemented by a quantum channel followed by an orthonormal measurement.