Quantum Information Theory, Spring 2019

Exercise Set 10 **in-class practice problems**

1. **Separable maps:**

   (a) Let $\Xi \in \mathrm{CP}(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z} \otimes \mathcal{W})$. Show that $\Xi \in \mathrm{SepCP}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W})$ if and only if there exist $A_a \in \mathrm{L}(\mathcal{X}, \mathcal{Z})$ and $B_a \in \mathrm{L}(\mathcal{Y}, \mathcal{W})$ such that

   $$\Xi(X) = \sum_{a \in \Sigma} (A_a \otimes B_a) X (A_a \otimes B_a)^*,$$

   for all $X \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$.

   (b) Let $\Xi_1 \in \mathrm{SepCP}(\mathcal{X}, \mathcal{U} : \mathcal{Y}, \mathcal{V})$ and $\Xi_2 \in \mathrm{SepCP}(\mathcal{U}, \mathcal{Z} : \mathcal{V}, \mathcal{W})$. Show that their composition is also separable:

   $$\Xi_2 \circ \Xi_1 \in \mathrm{SepCP}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W}).$$

2. **Examples of separable maps:** Show that the following maps jointly implemented by Alice and Bob are separable:

   (a) Alice and Bob share a random variable with probability distribution $p = (p_{a,b} : a \in \Sigma, b \in \Gamma)$, where Alice has the first register and Bob has the second one. Moreover, Alice has a quantum register A and Bob has a quantum register B. They both observe their halves of the random variable. If Alice's value is $a \in \Sigma$, she applies a channel Φ_a on her register A. Similarly, if Bob's value is $b \in \Gamma$, he applies a channel Ψ_b on his register B.

   (b) Alice has a register A that she measures in the standard basis. She sends the measurement outcome $a \in \Sigma$ to Bob who applies a channel Ψ_a on his register B.

   (c) Any LOCC channel.

3. **Instruments:** Recall that an instrument is a collection $\{\Phi_a : a \in \Sigma\} \subset \mathrm{CP}(\mathcal{X}, \mathcal{Y})$ such that $\sum_{a \in \Sigma} \Phi_a \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$. When applied to state $\rho \in \mathrm{D}(\mathcal{X})$, it produces outcome $a \in \Sigma$ with probability $\mathrm{Tr}[\Phi_a(\rho)]$ and changes the state to $\rho_a = \Phi_a(\rho) / \mathrm{Tr}[\Phi_a(\rho)]$. Show that any instrument can be implemented by a quantum channel followed by an orthonormal measurement.