Quantum Information Theory, Spring 2019

Exercise Set 12 **in-class practice problems**

1. **Maximally entangled states:** A pure state $|\psi\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ is *maximally entangled* if

$$\mathrm{Tr}_{\mathsf{Y}}[|\psi\rangle\langle\psi|] = \frac{I_{\mathsf{X}}}{\dim(\mathcal{X})} \qquad \text{and} \qquad \mathrm{Tr}_{\mathsf{X}}[|\psi\rangle\langle\psi|] = \frac{I_{\mathsf{Y}}}{\dim(\mathcal{Y})}.$$

   (a) Show that it must be the case that $\dim(\mathcal{X}) = \dim(\mathcal{Y})$.

   (b) Let $|\psi\rangle_{\mathsf{XY}}, |\psi'\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ be two maximally entangled states. Show that there exist local unitaries $U_{\mathsf{X}} \in \mathrm{U}(\mathcal{X})$ and $V_{\mathsf{Y}} \in \mathrm{U}(\mathcal{Y})$ such that $(U_{\mathsf{X}} \otimes V_{\mathsf{Y}})|\psi\rangle_{\mathsf{XY}} = |\psi'\rangle_{\mathsf{XY}}$.

   (c) Let $|\psi\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ and $|\phi\rangle_{\mathsf{ZW}} \in \mathcal{S}(\mathcal{Z} \otimes \mathcal{W})$ be maximally entangled. Show that $|\psi\rangle_{\mathsf{XY}} \otimes |\phi\rangle_{\mathsf{ZW}}$ is also maximally entangled with respect to the partition $\mathcal{X} \otimes \mathcal{Z} : \mathcal{Y} \otimes \mathcal{W}$.

   (d) Let $|\psi\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ be a maximally entangled state with $\dim(\mathcal{X}) = \dim(\mathcal{Y}) = d$, and let

$$\tau = \frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle}{\sqrt{2}}$$

   be the canonical two-qubit maximally entangled state. Show that an exact copy of $|\psi\rangle_{\mathsf{XY}}$ can be obtained from $\tau^{\otimes n}$ by LOCC, for some large enough n. What is the smallest value of n for which this holds?

2. **Fidelity and composition of channels:** Let $\tau_1 \in \mathrm{D}(\mathcal{X})$, $\sigma \in \mathrm{D}(\mathcal{Y})$, $\tau_2 \in \mathrm{D}(\mathcal{Z})$ be quantum states and let $\Phi \in \mathrm{C}(\mathcal{X}, \mathcal{Y})$ and $\Psi \in \mathrm{C}(\mathcal{Y}, \mathcal{Z})$ be quantum channels. Assuming that

$$\mathrm{F}\big(\Phi(\tau_1), \sigma\big) > 1 - \varepsilon, \qquad \mathrm{F}\big(\Psi(\sigma), \tau_2\big) > 1 - \varepsilon, \tag{1}$$

   for some $\varepsilon > 0$, show that

$$\mathrm{F}\big((\Psi \circ \Phi)(\tau_1), \tau_2\big) > 1 - 4\varepsilon, \tag{2}$$

   where $\Psi \circ \Phi$ denotes the composition of the two channels.

   Hint: Recall that you showed in Problem Set 6 that fidelity is monotonic under any quantum channel. Also, you can use the following inequality (which you will show in the homework): $\mathrm{F}(\rho_1, \sigma)^2 + F(\rho_2, \sigma)^2 \leq 1 + \mathrm{F}(\rho_1, \rho_2)$*, for any states* $\rho_1, \rho_2, \sigma \in \mathrm{D}(\mathcal{X})$.

3. **From any state to any other:** Let $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ and $\sigma \in \mathrm{D}(\mathcal{Z} \otimes \mathcal{W})$ be two arbitrary pure states. How many copies of the state σ can be distilled per copy of ρ?