Quantum Information Theory, Spring 2019

Exercise Set 14 **in-class practice problems**

1. **Quantum state merging:** Consider the pure state

$$|\Psi_{ABR}\rangle = |\phi^+_{A_1B_1}\rangle \otimes |\phi^+_{A_2R_1}\rangle \otimes |\phi^+_{B_2R_2}\rangle,$$

where $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Find a protocol for quantum state merging that achieves the rates discussed in class but uses only a single copy of $|\Psi_{ABR}\rangle$ at a time.

2. **Noisy teleportation:** In teleportation, Alice and Bob use maximally entangled states $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ as a resource in order to communicate qubits from Alice to Bob by sending only bits (for each maximally entangled state, they can communicate one qubit by sending 2 bits).

Now let $\rho_{AB} \in \mathrm{D}(\mathcal{A} \otimes \mathcal{B})$. Can Alice and Bob use ρ_{AB} as a resource state for teleportation? How many qubits can they communicate – and how many bits do they need to send to do so – per copy of the resource state ρ_{AB}?

Hint: Use quantum state merging. You may assume that Alice and Bob also have access to a supply of maximally entangled states, but they need to return them at the end of the protocol.

3. **Uhlmann's theorem:** Let $|\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$, $|\Phi\rangle \in \mathcal{X} \otimes \mathcal{Z}$ be purifications of $\rho, \sigma \in D(\mathcal{X})$, respectively, with $\dim \mathcal{Y} \leq \dim \mathcal{Z}$. Show that there exists an isometry $V_{Y \to Z} \colon \mathcal{Y} \to \mathcal{Z}$ such that

$$F(\rho, \sigma) = |\langle \Phi | I_X \otimes V_{Y \to Z} | \Psi \rangle|.$$

Hint: Use Uhlmann's theorem.

4. **Average:** Show that, for every operator $M_{AR} \in L(\mathcal{A} \otimes \mathcal{R})$, we have

$$\int (U_A^\dagger \otimes I_R) M_{AR} (U_A \otimes I_R) \, dU_A = \frac{I_A}{d_A} \otimes \mathrm{Tr}_A[M_{AR}]$$

and therefore, if $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$,

$$\int \mathrm{Tr}_{A_1}\left[(U_A^\dagger \otimes I_R) M_{AR} (U_A \otimes I_R)\right] dU_A = \frac{I_{A_2}}{d_{A_2}} \otimes \mathrm{Tr}_A[M_{AR}].$$

Can you interpret this equation in the context of the decoupling theorem?

5. **Partial trace and product measurements:** In class we used the following fact: For every state $\rho_{XY} \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ and measurement operators $0 \leq Q_X \leq I_X$ and $0 \leq Q_Y \leq I_Y$, we have

$$\mathrm{Tr}_Y[(Q_X \otimes Q_Y)\rho_{XY}(Q_X \otimes Q_Y)] \leq Q_X \rho_X Q_X$$

and therefore

$$\mathrm{Tr}\Big[\mathrm{Tr}_Y\left[(Q_X \otimes Q_Y)\rho_{XY}(Q_X \otimes Q_Y)\right]^2\Big] \leq \mathrm{Tr}\Big[(Q_X \rho_X Q_X)^2\Big].$$

Prove this.