Quantum Information Theory, Spring 2019

Exercise Set 2 **in-class practice problems**

Throughout, X, Y, Z denote quantum systems with Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$. The sets $\{|x\rangle\}$ and $\{|y\rangle\}$ denote arbitrary orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

1. **Computing reduced states:** Compute $\rho_X = \operatorname{tr}_Y[\rho]$ and $\rho_Y = \operatorname{tr}_X[\rho]$ in the following situations:

   (a) When $\rho = |\Psi\rangle\langle\Psi|$ is the two-qubit pure state given by
   $$|\Psi\rangle = \frac{1}{3}\big(|0,0\rangle + 2\,|0,1\rangle + 2\,|1,0\rangle\big) \in \mathcal{X}\otimes\mathcal{Y}$$
   and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^2$. If this calculation seems too painful to carry out, use (c) below.

   (b) A classical state $\rho = \sum_{x,y} p(x,y)\,|x,y\rangle\langle x,y|$ corresponding to an arbitrary joint probability distribution $p(x,y)$.

   Now consider a general pure state $\rho = |\Psi\rangle\langle\Psi|$ given in the form $|\Psi\rangle = \sum_{x,y} A_{x,y}\,|x\rangle\otimes|y\rangle \in \mathcal{X}\otimes\mathcal{Y}$.

   (c) Verify that $\rho_X = AA^*$ and $\rho_Y = A^T\bar{A}$.

   Solution. (a) By symmetry, $\rho_X = \rho_Y$, so it suffices to compute the former:
   $$\begin{aligned}\rho &= \frac{1}{9}\left(|0,0\rangle + 2\,|0,1\rangle + 2\,|1,0\rangle\right)\left(\langle 0,0| + 2\,\langle 0,1| + 2\,\langle 1,0|\right)\\ &= \frac{1}{9}\,|0,0\rangle\langle 0,0| + \frac{2}{9}\left(|0,0\rangle\langle 0,1| + |0,0\rangle\langle 1,0| + |0,1\rangle\langle 0,0| + |1,0\rangle\langle 0,0|\right)\\ &+ \frac{4}{9}\left(|0,1\rangle\langle 0,1| + |0,1\rangle\langle 1,0| + |1,0\rangle\langle 0,1| + |1,0\rangle\langle 1,0|\right)\end{aligned}$$
   Thus:
   $$\rho_X = \frac{1}{9}\,|0\rangle\langle 0| + \frac{2}{9}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) + \frac{4}{9}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{9}\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}$$
   We can also use part (c), which is much more efficient. The corresponding matrix is
   $$A = \frac{1}{\sqrt{9}}\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}$$
   and so
   $$\rho_X = AA^* = \frac{1}{9}\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 0\end{pmatrix} = \frac{1}{9}\begin{pmatrix}5 & 2\\ 2 & 4\end{pmatrix}.$$

   (b) Clearly,
   $$\rho_X = \operatorname{tr}_Y[\rho] = \sum_{x,y} p(x,y)\operatorname{tr}_Y[|x,y\rangle\langle x,y|] = \sum_{x,y} p(x,y)\,|x\rangle\langle x| = \sum_x p(x)\,|x\rangle\langle x|\,,$$
   where $p(x) = \sum_y p(x,y)$ is the marginal distribution of the first random variable. Similarly,
   $$\rho_Y = \sum_y p(y)\,|y\rangle\langle y|$$
   where $p(y) = \sum_x p(x,y)$.

(c) Since interchanging X and the Y amounts to transposing the matrix A, it suffices to establish the first formula. Indeed,

$$\rho = \sum_{x,y,x',y'} A_{x,y} |x,y\rangle \bar{A}_{x',y'} \langle x',y'| = \sum_{x,y,x',y'} A_{x,y} A^*_{y',x'} |x,y\rangle \langle x',y'|$$

and so

$$\begin{aligned}\rho_X = \operatorname{tr}_Y[\rho] &= \sum_{x,y,x',y'} A_{x,y} A^*_{y',x'} \operatorname{tr}_Y[|x,y\rangle \langle x',y'|] = \sum_{x,y,x',y'} A_{x,y} A^*_{y',x'} |x\rangle \langle x'| \delta_{y,y'} \\ &= \sum_{x,x',y'} A_{x,y} A^*_{y,x'} |x\rangle \langle x'| = \sum_{x,x'} (AA^*)_{x,x'} |x\rangle \langle x'| = AA^*.\end{aligned}$$

□

2. **Partial trace trickery:** Verify the following calculational rules for the partial trace. For all $A, B \in L(\mathcal{X})$, $M \in L(\mathcal{X} \otimes \mathcal{Y})$, and $N \in L(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$:

   (a) $\operatorname{tr}_Y[(A \otimes I_Y)M(B \otimes I_Y)] = A \operatorname{tr}_Y[M]B$,
   (b) $\operatorname{tr}_{XY}[N] = \operatorname{tr}_X[\operatorname{tr}_Y[N]] = \operatorname{tr}_Y[\operatorname{tr}_X[N]]$,

   What does the last rule look like if there is no Z-system?

   Solution. (a)

$$\begin{aligned}\operatorname{tr}_Y[(A \otimes I_Y)M(B \otimes I_Y)] &= \sum_y (I_X \otimes \langle y|)(A \otimes I_Y)M(B \otimes I_Y)(I_X \otimes |y\rangle) \\ = \sum_y (A \otimes \langle y|)M(B \otimes |y\rangle) &= \sum_y A(I_X \otimes \langle y|)M(I_X \otimes |y\rangle)B = A \operatorname{tr}_Y[M]B\end{aligned}$$

   (b) Note that $\operatorname{tr}_Y[N] \in L(\mathcal{X} \otimes \mathcal{Z})$, while $\operatorname{tr}_X[N] \in L(\mathcal{Y} \otimes \mathcal{Z})$.

$$\begin{aligned}\operatorname{tr}_X[\operatorname{tr}_Y[N]] &= \sum_x (\langle x| \otimes I_Z) \operatorname{tr}_Y[N](|x\rangle \otimes I_Z) \\ &= \sum_{x,y} (\langle x| \otimes I_Z)(I_X \otimes \langle y| \otimes I_Z)N(I_X \otimes |y\rangle \otimes I_Z)(|x\rangle \otimes I_Z) \\ &= \sum_{x,y} (\langle x| \otimes \langle y| \otimes I_Z)N(|x\rangle \otimes |y\rangle \otimes I_Z) = \operatorname{tr}_{XY}[N]\end{aligned}$$

   and similarly for $\operatorname{tr}_Y[\operatorname{tr}_X[N]]$.

   If there is no Z-system then (b) becomes

$$\operatorname{tr}[O] = \operatorname{tr}[\operatorname{tr}_Y[O]] = \operatorname{tr}[\operatorname{tr}_X[O]]$$

   for $O \in L(\mathcal{X} \otimes \mathcal{Y})$. □

3. **Schmidt decomposition:** In class, we discussed that any pure state $|\Psi\rangle = \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle$ has a Schmidt decomposition.

(a) Show that, if $\sum_i s_i |e_i\rangle \langle f_i|$ is a singular value decomposition of $A = \sum_{x,y} A_{x,y} |x\rangle \langle y|$, then $\sum_i s_i |e_i\rangle \otimes |\overline{f_i}\rangle$ is a Schmidt decomposition of $|\Psi\rangle$.

(b) Find a Schmidt decomposition of the following two-qubit pure state:

$$|\Psi\rangle = \frac{\sqrt{2}+1}{\sqrt{12}}(|00\rangle + |11\rangle) + \frac{\sqrt{2}-1}{\sqrt{12}}(|01\rangle + |10\rangle)$$

Solution. (a) We only need to verify the displayed formula (since it has the desired properties). For this, note that $|\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y}$ is obtained from $A \in L(\mathcal{Y}, \mathcal{X}) = \mathcal{X} \otimes \mathcal{Y}^*$ by transposing the second tensor factor (in the basis $|y\rangle$). But now we see that the displayed formula is obtained by the same operation from the formula for the SVD, since $|\overline{f_i}\rangle = \langle f_i|^T$.

If you are still sceptical, you can also verify the displayed formula by a direct calculation:

$$\begin{aligned} |\Psi\rangle &= \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle = \sum_{x,y} \langle x|A|y\rangle |x\rangle \otimes |y\rangle = \sum_i s_i \sum_{x,y} \langle x|e_i\rangle \langle f_i|y\rangle |x\rangle \otimes |y\rangle \\ &= \sum_i s_i \left(\sum_x |x\rangle \langle x|e_i\rangle\right)\left(\sum_y |y\rangle \langle f_i|y\rangle\right) = \sum_i s_i \left(\sum_x |x\rangle \langle x|e_i\rangle\right)\left(\sum_y |y\rangle \langle y|\overline{f_i}\rangle\right) \\ &= \sum_i s_i |e_i\rangle \otimes |\overline{f_i}\rangle \end{aligned}$$

(b) We need a singular value decomposition of the matrix

$$A = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{2}+1 & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2}+1 \end{pmatrix}.$$

Since this matrix is already Hermitian and positive (semi)definite, its singular value decomposition is the same as its eigendecompositions. But it is easy to see that

$$A = \frac{\sqrt{2}}{\sqrt{12}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \sqrt{\frac{2}{3}} |+\rangle \langle +| + \sqrt{\frac{1}{3}} |-\rangle \langle -|$$

and hence

$$|\Psi\rangle = \sqrt{\frac{2}{3}} |+,+\rangle + \sqrt{\frac{1}{3}} |-,-\rangle.$$

Here, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ as defined in Lecture 1.

$\square$

4. **Projective measurements:** Let $\mu\colon \Omega \to \mathrm{Pos}(\mathcal{X})$ be a *projective* measurement. Show that the projections $\mu(\omega)$ are pairwise orthogonal, i.e., $\mu(\omega)\mu(\omega') = 0$ for $\omega \neq \omega'$.

Solution. If P and Q are positive semidefinite operators then $\mathrm{tr}[PQ] \geq 0$, with equality if and only if $PQ = 0$. But

$$0 = \mathrm{tr}[\mu(\omega)] - \mathrm{tr}[\mu(\omega)^2] = \mathrm{tr}[\mu(\omega)(I - \mu(\omega))] = \sum_{\omega' \neq \omega} \underbrace{\mathrm{tr}[\mu(\omega)\mu(\omega')]}_{\geq 0}.$$

Thus, the summands must individually be zero, which in turn implies the claim (by the equality condition mentioned above). $\square$

5. **Observables (for physicists only):** In this problem we discuss the relationship between projective measurements and 'observables' as introduced in a basic quantum mechanics class. An *observable* on a quantum system X is by definition a Hermitian operator on $\mathcal{X}$.

   (a) Let $\mu\colon \Omega \to \mathrm{Pos}(\mathcal{X})$ be a *projective* measurement with $\Omega \subseteq \mathbb{R}$. Show that

   $$O = \sum_{\omega \in \Omega} \omega \mu(\omega) \tag{1}$$

   is an observable. Conversely, show that any observable can be written as in Eq. (1) for a suitable projective measurement μ.

   (b) What are the eigenvalues and eigenspaces of O?

   (c) Now suppose that the system is in state ρ and we perform the measurement μ. Show that the expectation value of the measurement outcome is given by $\mathrm{tr}[\rho O]$. For a pure state $\rho = |\psi\rangle\langle\psi|$, this can also be written as $\langle\psi|O|\psi\rangle$. Do you recognize these formulas from your quantum mechanics class?

   (d) Let Y be another quantum system. Show that $O \otimes I_Y$ is an observable on the joint system and argue that it is associated to the measurement $\mu \otimes I_Y$ defined in class.

Solution. (a) Since each $\omega \in \mathbb{R}$ and each $\mu(\omega)$ is Hermitian, it is clear that O is an observable. For the converse, note that if O is an observable then we can simply define Ω as the set of eigenvalue of O and the $\mu(w)$ as the corresponding eigenprojections. Indeed, since O is Hermitian it has a spectral decomposition

$$O = \sum_{i=1}^{r} \lambda_i |v_i\rangle\langle v_i|$$

where each $\lambda_i \in \mathbb{R}$ and the $\{|v_i\rangle\}$ are an orthonormal basis of $\mathcal{X}$. Define $\Omega = \{\lambda_i : i = 1, \dots, r\}$ and define the measurement

$$\mu\colon \Omega \to \mathrm{Pos}(\mathcal{X}), \quad \mu(x) = \sum_{i:\lambda_i = \omega} |v_i\rangle\langle v_i|.$$

Clearly, each $\mu(\omega)$ is a projection, and $\sum_{\omega\in\Omega} \mu(\omega) = I_X$ since the $\{|v_i\rangle\}$ form an orthonormal basis. Finally, we find:

$$\sum_{\omega\in\Omega} \omega\mu(\omega) = \sum_{\omega\in\Omega} \omega \sum_{i:\lambda_i=\omega} |v_i\rangle\langle v_i| = \sum_{\omega\in\Omega} \sum_{i:\lambda_i=\omega} \lambda_i |v_i\rangle\langle v_i| = \sum_{i:\lambda_i} \lambda_i |v_i\rangle\langle v_i| = O,$$

concluding the proof.

(b) It follows from problem 4 above that Eq. (1) is the spectral decomposition of O.

(c) We calculate the expectation value from its definition:

$$\sum_{\omega\in\Omega} \omega \Pr(\text{outcome } \omega) = \sum_{\omega\in\Omega} \omega \,\mathrm{tr}[\mu(w)\rho] = \mathrm{tr}[\sum_{\omega\in\Omega} \omega\mu(w)\rho] = \mathrm{tr}[O\rho].$$

If $\rho = |\psi\rangle\langle\psi|$ then as usual, by cyclicity of the trace,

$$\mathrm{tr}[O|\psi\rangle\langle\psi|] = \mathrm{tr}[\langle\psi|O|\psi\rangle] = \langle\psi|O|\langle\psi|\rangle.$$

(The second trace is the trace of a scalar.)

(d) For an observable in the form of Eq. (1),

$$O \otimes I = \sum_{\omega \in \Omega} \omega(\mu(\omega) \otimes I_Y),$$

where we note that $\mu \otimes I_Y$ is again a projective measurement.

□