Throughout, $X$, $Y$, $Z$ denote quantum systems with Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$. The sets $\{|x\rangle\}$ and $\{|y\rangle\}$ denote arbitrary orthonormal bases of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

1. **Computing reduced states:** Compute $\rho_X = \mathrm{tr}_Y [\rho]$ and $\rho_Y = \mathrm{tr}_X [\rho]$ in the following situations:

   (a) When $\rho = \ket{\Psi} \bra{\Psi}$ is the two-qubit pure state given by
   $$\ket{\Psi} = \frac{1}{3} \left( |0,0\rangle + 2 |0,1\rangle + 2 |1,0\rangle \right) \in \mathcal{X} \otimes \mathcal{Y}$$
   and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^2$. If this calculation seems too painful to carry out, use (c) below.

   (b) A classical state $\rho = \sum_{x,y} p(x,y) |x,y\rangle \langle x,y|$ corresponding to an arbitrary joint probability distribution $p(x,y)$.

   Now consider a general pure state $\rho = \ket{\Psi} \bra{\Psi}$ given in the form $\ket{\Psi} = \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle \in \mathcal{X} \otimes \mathcal{Y}$.

   (c) Verify that $\rho_X = AA^*$ and $\rho_Y = A^T \bar{A}$.

**Solution.** (a) By symmetry, $\rho_X = \rho_Y$, so it suffices to compute the former:

$$\rho = \frac{1}{9} \left( |0,0\rangle + 2 |0,1\rangle + 2 |1,0\rangle \right) \left( \langle 0,0| + 2 \langle 0,1| + 2 \langle 1,0| \right)$$

$$= \frac{1}{9} |0,0\rangle \langle 0,0| + \frac{2}{9} |0,1\rangle \langle 0,1| + \frac{2}{9} |1,0\rangle \langle 1,0| + \frac{4}{9} |0,1\rangle \langle 1,0| + \frac{4}{9} |1,0\rangle \langle 0,1|$$

Thus:

$$\rho_X = \frac{1}{9} |0\rangle \langle 0| + \frac{2}{9} |0\rangle \langle 1| + \frac{2}{9} |1\rangle \langle 0| + \frac{4}{9} |0\rangle \langle 1| + \frac{4}{9} |1\rangle \langle 1| = \frac{1}{9} \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}$$

We can also use part (c), which is much more efficient. The corresponding matrix is

$$A = \frac{1}{\sqrt{9}} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$$

and so

$$\rho_X = AA^* = \frac{1}{9} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}.$$ (b) Clearly,

$$\rho_X = \mathrm{tr}_Y [\rho] = \sum_{x,y} p(x,y) \mathrm{tr}_Y [|x,y\rangle \langle x,y|] = \sum_{x,y} p(x,y) |x\rangle \langle x| = \sum_x p(x) |x\rangle \langle x|,$$

where $p(x) = \sum_y p(x,y)$ is the marginal distribution of the first random variable. Similarly,

$$\rho_Y = \sum_y p(y) |y\rangle \langle y|$$

where $p(y) = \sum_x p(x,y)$. 


(c) Since interchanging $X$ and the $Y$ amounts to transposing the matrix $A$, it suffices to establish the first formula. Indeed,

$$
\rho = \sum_{x,y,x',y'} A_{x,y} |x,y\rangle \langle x',y'| = \sum_{x,y,x',y'} A_{x,y} A_{y',x'}^* |x,y\rangle \langle x',y'|
$$

and so

$$
\rho_X = \text{tr}_Y[\rho] = \sum_{x,y,x',y'} A_{x,y} A_{y',x'}^* \text{tr}_Y[|x,y\rangle \langle x',y'|] = \sum_{x,y,x',y'} A_{x,y} A_{y',x'}^* |x\rangle \langle x'| \delta_{y,y'}
$$

$$
= \sum_{x,x',y'} A_{x,y} A_{y',x'}^* |x\rangle \langle x'| = (AA^*)_{x,x'} |x\rangle \langle x'| = AA^*.
$$

\[\square\]

2. Partial trace trickery: Verify the following calculational rules for the partial trace. For all $A,B \in L(\mathcal{X})$, $M \in L(\mathcal{X} \otimes \mathcal{Y})$, and $N \in L(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$:

(a) $\text{tr}_Y[(A \otimes I_Y)M(B \otimes I_Y)] = A \text{tr}_Y[M]B$,

(b) $\text{tr}_{XY}[N] = \text{tr}_X[\text{tr}_Y[N]] = \text{tr}_Y[\text{tr}_X[N]]$.

What does the last rule look like if there is no $Z$-system?

**Solution.** (a)

$$
\text{tr}_Y[(A \otimes I_Y)M(B \otimes I_Y)] = \sum_y (I_X \otimes \langle y|)(A \otimes I_Y)M(B \otimes I_Y)(I_X \otimes |y\rangle)
$$

$$
= \sum_y (A \otimes \langle y|)M(B \otimes |y\rangle) = \sum_y A(I_X \otimes \langle y|)M(I_X \otimes |y\rangle)B = A \text{tr}_Y[M]B
$$

(b) Note that $\text{tr}_Y[N] \in L(\mathcal{X} \otimes \mathcal{Z})$, while $\text{tr}_X[N] \in L(\mathcal{Y} \otimes \mathcal{Z})$.

$$
\text{tr}_X[\text{tr}_Y[N]] = \sum_x (|x\rangle \otimes I_Z) \text{tr}_Y[N] (|x\rangle \otimes I_Z)
$$

$$
= \sum_{x,y} (|x\rangle \otimes I_Z)(I_X \otimes |y\rangle \otimes I_Z)N(I_X \otimes |y\rangle \otimes I_Z)(|x\rangle \otimes I_Z)
$$

$$
= \sum_{x,y} (|x\rangle \otimes (|y\rangle \otimes I_Z)N(|x\rangle \otimes |y\rangle \otimes I_Z) = \text{tr}_{XY}[N]
$$

and similarly for $\text{tr}_Y[\text{tr}_X[N]]$.

If there is no $Z$-system then (b) becomes

$$
\text{tr}[O] = \text{tr}[\text{tr}_Y[O]] = \text{tr}[\text{tr}_X[O]]
$$

for $O \in L(\mathcal{X} \otimes \mathcal{Y})$.

\[\square\]

3. Schmidt decomposition: In class, we discussed that any pure state $|\Psi\rangle = \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle$ has a Schmidt decomposition.
(a) Show that, if \( \sum_i s_i |e_i\rangle \langle f_i| \) is a singular value decomposition of \( A = \sum_{x,y} A_{x,y} |x\rangle \langle y| \), then \( \sum_i s_i |e_i\rangle \otimes |f_i\rangle \) is a Schmidt decomposition of \( |\Psi\rangle \).

(b) Find a Schmidt decomposition of the following two-qubit pure state:
\[
|\Psi\rangle = \sqrt{\frac{2}{3}} (|00\rangle + |11\rangle) + \sqrt{\frac{1}{3}} (|01\rangle + |10\rangle)
\]

**Solution.** (a) We only need to verify the displayed formula (since it has the desired properties). For this, note that \( |\Psi\rangle \in \mathcal{X} \otimes \mathcal{Y} \) is obtained from \( A \in L(\mathcal{Y}, \mathcal{X}) = \mathcal{X} \otimes \mathcal{Y}^* \) by transposing the second tensor factor (in the basis \( |y\rangle \)). But now we see that the displayed formula is obtained by the same operation from the formula for the SVD, since \( |f_i\rangle = \langle f_i| \). If you are still sceptical, you can also verify the displayed formula by a direct calculation:

\[
|\Psi\rangle = \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle = \sum_{x,y} \langle x| A |y\rangle |x\rangle \otimes |y\rangle = \sum_i s_i \sum_{x,y} \langle x| e_i \rangle \langle f_i| |y\rangle |x\rangle \otimes |y\rangle
\]

\[
= \sum_i s_i \left( \sum_x |x\rangle \langle x| e_i \rangle \right) \left( \sum_y |y\rangle \langle f_i| y\rangle \right) = \sum_i s_i \left( \sum_x |x\rangle \langle x| e_i \rangle \right) \left( \sum_y |y\rangle \langle y| f_i \rangle \right) = \sum_i s_i |e_i\rangle \otimes |f_i\rangle
\]

(b) We need a singular value decomposition of the matrix
\[
A = \frac{1}{\sqrt{12}} \begin{pmatrix} \sqrt{2} + 1 & \sqrt{2} - 1 \\ \sqrt{2} - 1 & \sqrt{2} + 1 \end{pmatrix}
\]

Since this matrix is already Hermitian and positive (semi)definite, its singular value decomposition is the same as its eigendecompositions. But it is easy to see that
\[
A = \frac{\sqrt{2}}{\sqrt{12}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{12}} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \sqrt{\frac{2}{3}} |+\rangle \langle +| + \sqrt{\frac{1}{3}} |-\rangle \langle -|
\]

and hence
\[
|\Psi\rangle = \sqrt{\frac{2}{3}} |+\rangle + \sqrt{\frac{1}{3}} |-\rangle.
\]

Here, \( |\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle) \) as defined in Lecture 1.

\(\square\)

4. **Projective measurements:** Let \( \mu : \Omega \to \text{Pos}(\mathcal{X}) \) be a projective measurement. Show that the projections \( \mu(\omega) \) are pairwise orthogonal, i.e., \( \mu(\omega) \mu(\omega') = 0 \) for \( \omega \neq \omega' \).

**Solution.** If \( P \) and \( Q \) are positive semidefinite operators then \( \text{tr}[PQ] \geq 0 \), with equality if and only if \( PQ = 0 \). But
\[
0 = \text{tr}[\mu(\omega)] - \text{tr}[\mu(\omega)^2] = \text{tr}[\mu(\omega)(I - \mu(\omega))] = \sum_{\omega' \neq \omega} \text{tr}[\mu(\omega)\mu(\omega')] \geq 0.
\]

Thus, the summands must individually be zero, which in turn implies the claim (by the equality condition mentioned above).

\(\square\)
5. **Observables (for physicists only):** In this problem we discuss the relationship between projective measurements and ‘observables’ as introduced in a basic quantum mechanics class. An observable on a quantum system $X$ is by definition a Hermitian operator on $X$.

(a) Let $\mu: \Omega \to \text{Pos}(X)$ be a projective measurement with $\Omega \subseteq \mathbb{R}$. Show that

$$O = \sum_{\omega \in \Omega} \omega \mu(\omega) \tag{1}$$

is an observable. Conversely, show that any observable can be written as in Eq. (1) for a suitable projective measurement $\mu$.

(b) What are the eigenvalues and eigenspaces of $O$?

(c) Now suppose that the system is in state $\rho$ and we perform the measurement $\mu$. Show that the expectation value of the measurement outcome is given by $\text{tr}[\rho O]$. For a pure state $\rho = \ket{\psi} \bra{\psi}$, this can also be written as $\bra{\psi} O \ket{\psi}$. Do you recognize these formulas from your quantum mechanics class?

(d) Let $Y$ be another quantum system. Show that $O \otimes I_Y$ is an observable on the joint system and argue that it is associated to the measurement $\mu \otimes I_Y$ defined in class.

**Solution.** (a) Since each $\omega \in \mathbb{R}$ and each $\mu(\omega)$ is Hermitian, it is clear that $O$ is an observable. For the converse, note that if $O$ is an observable then we can simply define $\Omega$ as the set of eigenvalue of $O$ and the $\mu(\omega)$ as the corresponding eigenprojections. Indeed, since $O$ is Hermitian it has a spectral decomposition

$$O = \sum_{i=1}^{r} \lambda_i \ket{v_i} \bra{v_i}$$

where each $\lambda_i \in \mathbb{R}$ and the $\{\ket{v_i}\}$ are an orthonormal basis of $X$. Define $\Omega = \{\lambda_i : i = 1, \ldots, r\}$ and define the measurement

$$\mu: \Omega \to \text{Pos}(X), \quad \mu(x) = \sum_{i : \lambda_i = x} \ket{v_i} \bra{v_i}.$$ 

Clearly, each $\mu(\omega)$ is a projection, and $\sum_{\omega \in \Omega} \mu(\omega) = I_X$ since the $\{\ket{v_i}\}$ form an orthonormal basis. Finally, we find:

$$\sum_{\omega \in \Omega} \omega \mu(\omega) = \sum_{\omega \in \Omega} \omega \sum_{i : \lambda_i = \omega} \ket{v_i} \bra{v_i} = \sum_{\omega \in \Omega} \sum_{i : \lambda_i = \omega} \lambda_i \ket{v_i} \bra{v_i} = \sum_{i : \lambda_i} \lambda_i \ket{v_i} \bra{v_i} = O,$$

concluding the proof.

(b) It follows from problem 4 above that Eq. (1) is the spectral decomposition of $O$.

(c) We calculate the expectation value from its definition:

$$\sum_{\omega \in \Omega} \omega \Pr(\text{outcome } \omega) = \sum_{\omega \in \Omega} \omega \text{tr}[\mu(\omega) \rho] = \text{tr}\left[\sum_{\omega \in \Omega} \omega \mu(\omega) \rho\right] = \text{tr}[O \rho].$$

If $\rho = \ket{\psi} \bra{\psi}$ then as usual, by cyclicity of the trace,

$$\text{tr}[O \ket{\psi} \bra{\psi}] = \text{tr}\left[\bra{\psi} O \ket{\psi}\right] = \bra{\psi} O \ket{\psi}.$$

(The second trace is the trace of a scalar.)
(d) For an observable in the form of Eq. (1),

\[ O \otimes I = \sum_{\omega \in \Omega} \omega (\mu(\omega) \otimes I_Y), \]

where we note that \( \mu \otimes I_Y \) is again a projective measurement.