Quantum Information Theory, Spring 2019

Exercise Set 4 **in-class practice problems**

Throughout, X, Y, Z denote quantum systems with complex Euclidean spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$.

1. **Positive semidefinite operators:** Let $X \in \mathrm{L}(\mathcal{X})$ where $\mathcal{X} = \mathbb{C}^{\Sigma}$. The following are all equivalent to X being positive semidefinite:

   - $P = Y^*Y$, for some $Y \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$.
   - $\langle\psi|P|\psi\rangle \geq 0$, for all $|\psi\rangle \in \mathcal{X}$.
   - $P = U\,\mathrm{diag}(\lambda_1, \ldots, \lambda_n)U^*$, for some $U \in \mathrm{U}(\mathcal{X})$ and some $\lambda_i \geq 0$.
   - There exists a set of vectors $\{|v_a\rangle \in \mathcal{X} : a \in \Sigma\}$ such that, for all $a, b \in \Sigma$, $P_{a,b} = \langle v_a|v_b\rangle$.
   - $\mathrm{Tr}(PQ) \geq 0$, for all $Q \in \mathrm{Pos}(\mathcal{X})$.

   Can you see why these characterizations are equivalent? Use these different characterizations to show that

   (a) Any positive semidefinite operator is Hermitian.
   (b) Any convex combination of positive semidefinite operators is positive semidefinite.
   (c) If a positive semidefinite operator is invertible, its inverse is again positive semidefinite.
   (d) If $P \in \mathrm{Pos}(\mathcal{X})$ and $A \in \mathrm{L}(\mathcal{X})$, then A^*PA is positive semidefinite.
   (e) If P is positive semidefinite, then so is $\sqrt{P}$.

2. **The completely dephasing channel:** Let $\Delta \in \mathrm{T}(\mathcal{X})$ be the completely dephasing map on $\mathcal{X} = \mathbb{C}^{\Sigma}$.

   (a) Compute the output state when Δ is applied to one register of a maximally entangled state $|\Psi\rangle = \frac{1}{\sqrt{|\sigma|}} \sum_{a\in\Sigma} |a\rangle \otimes |a\rangle$.
   (b) Show that Δ is a quantum channel.

3. **Partial measurement:** Assume you have a two-qubit system in the following state:

$$|\psi\rangle = \frac{1}{\sqrt{30}}\big(|00\rangle + 2i|01\rangle - 3|10\rangle - 4i|11\rangle\big).$$

   (a) Assume you measure the second qubit in the standard basis. Compute the probabilities $p(0)$ and $p(1)$ of the two measurement outcomes.
   (b) If this measurement produced outcome 1, what is the state of the first qubit?

4. **Channels:**

   (a) Let $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ and $\Psi \in \mathrm{T}(\mathcal{Y}, \mathcal{Z})$. Show that if Φ and Ψ are quantum channels, then their composition $\Psi \circ \Phi$ is again a quantum channel.
   (b) Recall that we can define an inner product on $\mathrm{L}(\mathcal{X})$ by $\langle A, B\rangle = \mathrm{Tr}(A^*B)$. Define the adjoint of a superoperator $\Phi \in \mathrm{T}(\mathcal{X}, \mathcal{Y})$ as a superoperator $\Phi^* \in \mathrm{T}(\mathcal{Y}, \mathcal{X})$ such that $\langle\Phi^*(A), B\rangle = \langle A, \Phi(B)\rangle$, for all $A \in \mathrm{L}(\mathcal{Y})$ and $B \in \mathrm{L}(\mathcal{X})$. Show that Φ is a quantum channel if and only if Φ^* is unital (that is, $\Phi^*(I_{\mathsf{Y}}) = I_{\mathsf{X}}$) and completely positive.

5. **Linear probability assignments are measurements:** Let Σ be an alphabet, and let $p : \mathrm{Herm}(\mathcal{X}) \to \mathbb{R}^{\Sigma}$ be a linear function. Show that the following statements are equivalent:

   (a) $p(\rho)$ is a probability distribution on Σ for every $\rho \in \mathrm{D}(\mathcal{X})$.

   (b) There exists a measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ such that

   $$(p(H))(a) = \mathrm{Tr}(\mu(a)H)$$

   for all $H \in \mathrm{Herm}(\mathcal{X})$ and $a \in \Sigma$.