Quantum Information Theory, Spring 2019

Exercise Set 7 **in-class practice problems**

1. **Shannon entropy inequalities:** In this problem you will see that the Shannon entropy of probability distributions is more constrained than the entropy of general quantum states. Let $p_{XY} \in \mathcal{P}(\Sigma \times \Gamma)$ be a joint probability distribution. The entropies of p_{XY} and its marginal distributions p_X and p_Y are denoted by $H(XY)$, $H(X)$, and $H(Y)$, respectively.

   (a) Show that p_{XY} can be decomposed as $p_{XY}(x,y) = p_X(x)p_{Y|X=x}(y)$, where $p_{Y|X=x}$ is a probability distribution for each $x \in \Sigma$.

   (b) Deduce the following formula:

   $$H(XY) = H(X) + \sum_{x\in\Sigma} p_X(x) H(p_{Y|X=x})$$

   (c) Conclude that the Shannon entropy satifies the *monotonicity* inequality $H(XY) \geq H(X)$. Show that equality holds if and only if p_{XY} is of the form $p_{XY}(x,y) = p_X(x)\delta_{f(x),y}$ for some function $f\colon \Sigma \to \Gamma$ (i.e., the second random variable is a function of the first).

   (d) Conclude that $I(X:Y) \leq \min\{H(X), H(Y)\} \leq \log \min\{|\Sigma|, |\Gamma|\}$.

2. **Entropy of classical-quantum states:** Let $\{p_x, \rho_x\}$ be an ensemble, i.e., $p \in \mathcal{P}(\Sigma)$ and $\rho_x \in D(\mathcal{Y})$ for $x \in \Sigma$ and consider the corresponding *classical-quantum (cq) state*

   $$\rho_{XY} = \sum_{x\in\Sigma} p_x |x\rangle\langle x| \otimes \rho_x.$$

   (a) Prove the following formula that we used in the lecture:

   $$H(XY) = H(p) + \sum_{x\in\Sigma} p_x H(\rho_x)$$

   (b) Conclude that $H(XY) \geq H(X)$. When does equality hold?

   It is also true that $H(XY) \geq H(Y)$, but this requires a different argument (note that the situation is *not* symmetric since, unlike in Problem 1, system Y is not necessarily classical).

3. **Weak monotonicity:** For general quantum states, it is *not* true that $H(XY) \geq H(X)$ or, equivalently, that $I(X:Y) \leq H(Y)$.

   (a) Find a quantum state ρ_{XY} such that $H(XY) < H(X)$.

   However, the quantum entropy satisfies a weaker inequality, known as *weak monotonicity*: For every state $\rho_{XYZ} \in D(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$, it holds that

   $$H(XY) + H(YZ) \geq H(X) + H(Z).$$

   (b) Show that this inequality follows from the strong subadditivity inequality discussed in class.

   Hint: Use a purification.