Exercise Set 8 in-class practice problems

Quantum Information Theory, Spring 2019

1. Warmup:
   
   (a) Show that, if $\rho$ and $\sigma$ are both pure states, $D(\rho\|\sigma) \in \{0, \infty\}$.
   
   (b) Find a state $\rho$ and a channel $\Phi$ such that $H(\Phi[\rho]) < H(\rho)$.
   
   (c) Compute the relative entropy $D(\rho\|\sigma)$ for $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$ and $\sigma = \frac{1}{3}|+\rangle\langle +| + \frac{2}{3}|-\rangle\langle -|$. 

2. Matrix logarithm: Recall that the logarithm of a positive definite operator with eigendecomposition $Q = \sum_i \lambda_i |e_i\rangle\langle e_i|$ is defined as $\log(Q) = \sum_i \log(\lambda_i) |e_i\rangle\langle e_i|$ (as always, our logarithms are to base 2). Verify the following properties:

   (a) $\log(cI) = \log(c)I$ for every $c \geq 0$.
   
   (b) $\log(Q \otimes R) = \log(Q) \otimes I_Y + I_X \otimes \log(R)$ for all positive definite operators $Q$ on $\mathcal{X}$, $R$ on $\mathcal{Y}$.
   
   (c) $\log(\sum_{x \in \Sigma} p_x |x\rangle\langle x| \otimes \rho_x) = \sum_{x \in \Sigma} \log(p_x) |x\rangle\langle x| \otimes I_Y + \sum_{x \in \Sigma} |x\rangle\langle x| \otimes \log(\rho_x)$ for every ensemble $\{p_x, \rho_x\}_{x \in \Sigma}$ of positive definite operators $\rho_x \in D(\mathcal{Y})$.

   Warning: It is in general not true that $\log(QR) = \log(Q) + \log(R)$!

3. From Relative Entropy to Entropy and Mutual Information: Use Problem 1 to verify the following claims from class:

   (a) $D(\rho\|I_\mathcal{X}) = \log d - H(\rho)$ for every $\rho \in D(\mathcal{X})$, where $d = \dim \mathcal{X}$.
   
   (b) $D(\rho_{XY}\|\rho_X \otimes \rho_Y) = I(X : Y)_{\rho_{XY}}$ for every $\rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$, where $\rho_X = \text{Tr}_Y[\rho_{XY}]$ and $\rho_Y = \text{Tr}_X[\rho_{XY}]$. You may assume that all three operators are positive definite.

4. Entropy and ensembles: In this problem, you will prove the upper bound on the Holevo information that we discussed in class: For every ensemble $\{p_x, \rho_x\}$,

   $\chi(\{p_x, \rho_x\}) \leq H(p)$ or, equivalently, $H(\sum_x p_x \rho_x) \leq H(p) + \sum_x p_x H(\rho_x)$.

Moreover, equality holds if and only if the $\rho_x$ with $p_x > 0$ have pairwise orthogonal image.

   (a) First prove these claims assuming that each $\rho_x$ is a pure state, i.e., $\rho_x = |\psi_x\rangle\langle \psi_x|$.

   Hint: Consider the pure state $|\Phi\rangle = \sum_x \sqrt{p_x} |x\rangle \otimes |\psi_x\rangle$ and compare the entropy of the first system before and after measuring in the standard basis.

   (b) Now prove the claims for general $\rho_x$.

   Hint: Apply part (a) to a suitable ensemble obtained from the eigendecompositions of the $\rho_x$.

In terms of the cq-state corresponding to the ensemble, the above inequality can also be written as $H(XY) \geq H(Y)$. This confirms a claim made in Problem 2 of last week’s exercise set.