Lecture 10: Separable maps and LOCC

(Chapters 6.1.2 and 6.1.3.)

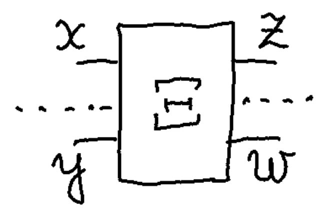
<u>Entanglement</u> = quantum correlations

Quantum correlations between two registers cannot be increased by
- <u>local operations</u>
- <u>classical communication</u> between these two registers.

Entanglement is defined as the resource that cannot be increased by LOCC (local operations and classical communication).

Mathematically, LOCC is hard to deal with, so we often relax it to a slightly larger class of separable operations:

<u>Def 6.17</u>: A completely positive map $\Xi \in CP(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z} \otimes \mathcal{W})$ is <u>separable</u> if



$$\Xi = \sum_{a \in \Sigma} \Phi_a \otimes \Psi_a$$

for some $\Phi_a \in CP(\mathcal{X}, \mathcal{Z})$ and $\Psi_a \in CP(\mathcal{Y}, \mathcal{W})$.

Notation: $\mathrm{SepCP}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W})$.

- A CP map Ξ is separable iff it has a Kraus representation with product Kraus operators:

$$\Xi(X) = \sum_{a\in\Sigma} (A_a \otimes B_a) X (A_a \otimes B_a)^* \qquad \text{(Prop. 6.18)}$$

- SepCP is closed under composition (Prop. 6.19)

Trace-preserving separable maps are called separable channels:

<u>Def. 6.20</u>: The set of <u>separable channels</u> is

$$\mathrm{SepC}(\mathcal{X},\mathcal{Z}:\mathcal{Y},\mathcal{W}) = \mathrm{SepCP}(\mathcal{X},\mathcal{Z}:\mathcal{Y},\mathcal{W}) \cap C(\mathcal{X}\otimes\mathcal{Y}, \mathcal{Z}\otimes\mathcal{W}).$$

A CP map is separable iff its Choi representation is:

<u>Prop. 6.22</u>: Let $\Xi \in CP(\mathcal{X}\otimes\mathcal{Y}, \mathcal{Z}\otimes\mathcal{W})$. Then $\Xi \in \mathrm{SepCP}(\mathcal{X},\mathcal{Z}:\mathcal{Y},\mathcal{W})$ iff $V J(\Xi) V^* \in \mathrm{Sep}(\mathcal{Z}\otimes\mathcal{X} : \mathcal{W}\otimes\mathcal{Y})$ where $V|z\rangle|w\rangle|x\rangle|y\rangle = |z\rangle|x\rangle|w\rangle|y\rangle$, for all $x\in\mathcal{X}, y\in\mathcal{Y}, z\in\mathcal{Z}, w\in\mathcal{W}$.

<u>Proof</u>: By Prop 6.18, $\Xi(X) = \sum_{a\in\Sigma} (A_a\otimes B_a) X (A_a\otimes B_a)^*$, so the Choi representation of Ξ is $J(\Xi) = \sum_{a\in\Sigma} \mathrm{vec}(A_a\otimes B_a)\,\mathrm{vec}(A_a\otimes B_a)^*$.

Since $V\,\mathrm{vec}(|z\rangle\langle x|\otimes|w\rangle\langle y|) = V\,\mathrm{vec}(|z\rangle|w\rangle\langle x|\langle y|)$
$= V|z\rangle|w\rangle|x\rangle|y\rangle = |z\rangle|x\rangle|w\rangle|y\rangle = \mathrm{vec}(|z\rangle\langle x|)\otimes\mathrm{vec}(|w\rangle\langle y|)$,
$V\,\mathrm{vec}(A_a\otimes B_a) = \mathrm{vec}(A_a)\otimes\mathrm{vec}(B_a)$ by linearity. Thus

$$V J(\Xi) V^* = \sum_{a\in\Sigma} \mathrm{vec}(A_a)\mathrm{vec}(A_a)^* \otimes \mathrm{vec}(B_a)\mathrm{vec}(B_a)^* \in \mathrm{Sep}(\mathcal{Z}\otimes\mathcal{X} : \mathcal{W}\otimes\mathcal{Y}).$$

Other direction: reverse the argument. $\square$

Entanglement rank

<u>Def. 6.14</u>: $R \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y}) \subseteq \mathrm{Pos}(\mathcal{X}\otimes\mathcal{Y})$ if
$R = \sum_{a\in\Sigma} \mathrm{vec}(A_a)\,\mathrm{vec}(A_a)^*$ for some $A_a \in L(\mathcal{Y},\mathcal{X})$ such that $\mathrm{rank}(A_a) \leq r$, for each $a\in\Sigma$. The <u>entanglement rank</u> of $R \in \mathrm{Pos}(\mathcal{X}\otimes\mathcal{Y})$ is the smallest r such that $R \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$.

Note that $\mathrm{Sep} = \mathrm{Ent}_1 \subset \cdots \subset \mathrm{Ent}_r \subset \mathrm{Ent}_{r+1} \subset \cdots \mathrm{Ent}_n = \mathrm{Pos}$ where $n = \min\{\dim\mathcal{X}, \dim\mathcal{Y}\}$ and all inclusions are strict. Separable channels cannot create entangled states from separable ones. In fact, they cannot increase the entanglement rank.

<u>Thm 6.23</u>: If $\Xi \in \mathrm{SepCP}(\mathcal{X},\mathcal{Z}:\mathcal{Y},\mathcal{W})$ and $P \in \mathrm{Ent}_r(\mathcal{X}:\mathcal{Y})$ then $\Xi(P) \in \mathrm{Ent}_r(\mathcal{Z}:\mathcal{W})$.

<u>Proof</u>: We can write $P = \sum_{b\in\Gamma} \mathrm{vec}(X_b)\,\mathrm{vec}(X_b)^*$, for some $X_b \in L(\mathcal{Y},\mathcal{X})$ such that $\mathrm{rank}(X_b) \leq r$. By Prop. 6.18, there exist $A_a \in L(\mathcal{X},\mathcal{Z})$ and $B_a \in L(\mathcal{Y},\mathcal{W})$ such that

$$\Xi(P) = \sum_{a\in\Sigma}\sum_{b\in\Gamma} (A_a\otimes B_a)\,\mathrm{vec}(X_b)\,\mathrm{vec}(X_b)^*(A_a\otimes B_a)^*$$
$$= \sum_{a\in\Sigma}\sum_{b\in\Gamma} \mathrm{vec}(A_a X_b B_a^T)\,\mathrm{vec}(A_a X_b B_a^T)^*.$$

Since $\mathrm{rank}(A_a X_b B_a^T) \leq \mathrm{rank}(X_b) \leq r$,

$$\Xi(P) \in \mathrm{Ent}_r(\mathcal{Z}:\mathcal{W}).$$ □

<u>Cor 6.24</u>:
If $\Xi \in \mathrm{SepCP}(\mathcal{X},\mathcal{Z}:\mathcal{Y},\mathcal{W})$ and $P \in \mathrm{Sep}(\mathcal{X}:\mathcal{Y})$ then $\Xi(P) \in \mathrm{Sep}(\mathcal{Z}:\mathcal{W})$.

Instruments

The most general operation that produces a classical outcome as well as a leftover quantum state is known as an instrument.

Def (see page 112): An instrument is a collection $\{\Phi_a : a\in\Sigma\} \subset CP(\mathcal{X},\mathcal{Y})$ such that $\sum_{a\in\Sigma} \Phi_a \in C(\mathcal{X},\mathcal{Y})$.

When applied to state $\rho\in D(\mathcal{X})$, it produces outcome $a\in\Sigma$ with probability $\mathrm{Tr}[\Phi_a(\rho)]$ and replaces the state by

$$\rho_a = \Phi_a(\rho)/\mathrm{Tr}[\Phi_a(\rho)] \in D(\mathcal{Y}).$$

Any instrument can be implemented by a quantum channel, followed by an orthonormal measurement.

LOCC channels

Local operations and classical communication can implement the following set of quantum channels.

Def 6.25 • One-way right LOCC channels are of the form

$\boxed{\text{Alice} \xrightarrow{a} \text{Bob}}$

$$\Xi = \sum_{a\in\Sigma} \Phi_a \otimes \Psi_a \in C(\mathcal{X}\otimes\mathcal{Y}, \mathcal{Z}\otimes\mathcal{W}) \text{ where}$$

$\{\Phi_a : a\in\Sigma\} \subset CP(\mathcal{X},\mathcal{Z})$ is an instrument and $\Psi_a \in C(\mathcal{Y},\mathcal{W})$.

• One-way left LOCC are similar, except $\Phi_a \in C(\mathcal{X},\mathcal{Z})$ and $\{\Psi_a : a\in\Sigma\}\subset CP(\mathcal{Y},\mathcal{W})$ is an instrument.

• $LOCC(\mathcal{X},\mathcal{Z} : \mathcal{Y},\mathcal{W})$ consists of finite compositions of these.

Intuitively, in a one-way left LOCC protocol Alice performs a local channel followed by a measurement. She then sends the outcome $a \in \Sigma$ to Bob. Depending on a, Bob applies a channel Ψ_a. Right LOCC protocols are similar, except the communication is from Bob to Alice and Alice's channel depends on Bob's measurement outcome.

Prop. 6.26: $\mathrm{LOCC}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W}) \subseteq \mathrm{SepC}(\mathcal{X}, \mathcal{Z} : \mathcal{Y}, \mathcal{W})$.

Separable and LOCC measurements (6.1.3)

Def 6.27: Let $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ be a measurement on registers X and Y. Let $\mathcal{Z} = \mathbb{C}^\Sigma$, $\mathcal{W} = \mathbb{C}^\Sigma$ and

$$\Phi_\mu \in C(\mathcal{X} \otimes \mathcal{Y}, \mathcal{Z} \otimes \mathcal{W})$$

be a quantum-to-classical channel defined as

$$\Phi_\mu(X) = \sum_{a \in \Sigma} \mathrm{Tr}\left[\mu(a) X\right] \cdot |a\rangle\langle a|_{\mathcal{Z}} \otimes |a\rangle\langle a|_{\mathcal{W}}$$

The measurement μ is separable / LOCC if Φ_μ is separable / LOCC.

A measurement is separable iff each measurement operator is separable.

Prop. 6.28 A measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is separable iff $\mu(a) \in \mathrm{Sep}(\mathcal{X} : \mathcal{Y})$ for each $a \in \Sigma$.

One-way LOCC measurements

<u>Def. 6.29</u>: A measurement $\mu \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$ is <u>one-way right LOCC</u> if there exists a measurement $\nu : \Gamma \to \mathrm{Pos}(\mathcal{X})$ on Alice's side and, for each $b \in \Gamma$, a measurement $\pi_b : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ on Bob's side such that

$$\mu(a) = \sum_{b \in \Gamma} \nu(b) \otimes \pi_b(a)$$

for every $a \in \Sigma$. Similarly, a measurement is <u>one-way left LOCC</u> if

$$\mu(a) = \sum_{b \in \Gamma} \pi_b(a) \otimes \nu(b)$$

where the roles of Alice and Bob are exchanged.

While one-way LOCC measurements may seem rather limited, they can perfectly discriminate any two orthogonal pure bipartite states, even if they are entangled.

<u>Thm 6.32</u>: If $|\psi_0\rangle, |\psi_1\rangle \in S(\mathcal{X} \otimes \mathcal{Y})$ are orthogonal then there exists a one-way LOCC measurement

$$\mu : \{0,1\} \to \mathrm{Pos}(\mathcal{X} \otimes \mathcal{Y})$$

such that $\langle \psi_0 | \mu(0) | \psi_0 \rangle = \langle \psi_1 | \mu(1) | \psi_1 \rangle = 1$.