Lecture 4: Measurements (Chapter 2.3)

Positive semidefinite operators
(see page 20 of Watrous book)

We call $P \in \mathrm{L}(\mathcal{X})$ <u>positive semidefinite</u>, $P \in \mathrm{Pos}(\overset{\mathbb{C}^{\Sigma}}{\overset{\shortparallel}{\mathcal{X}}})$, if any of the following equivalent conditions hold:

1. $P = Y^* Y$, for some $Y \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$ and some space $\mathcal{Y}$.
2. $\langle \psi | P | \psi \rangle \geq 0$, for all $|\psi\rangle \in \mathcal{X}$.
3. $P = U \operatorname{diag}(\lambda_1, \dots, \lambda_n) U^*$, for some $U \in \mathrm{U}(\mathcal{X})$ and $\lambda_i \geq 0$.
4. There exists a set of vectors $\{ |v_a\rangle \in \mathcal{X} : a \in \Sigma \}$ such that
$$P_{a,b} = \langle v_a | v_b \rangle.$$
not necessarily unit vectors
5. $\mathrm{Tr}(PQ) \geq 0$, for all $Q \in \mathrm{Pos}(\mathcal{X})$.

You can try to prove their equivalence as an exercise. Here are some simple consequences:

If $P \geq 0$ then

- $cP \geq 0$, for any $c \in \mathbb{R}$, $c \geq 0$
- $P + Q \geq 0$, for any $Q \in \mathrm{Pos}(\mathcal{X})$
- $P \otimes Q \geq 0$, for any $Q \in \mathrm{Pos}(\mathcal{Y})$

The completely dephasing channel $\Delta \in T(\mathcal{X})$

$\overset{\shortparallel}{\mathbb{C}^\Gamma}$

It preserves the diagonal entries and replaces all off-diagonal entries by zero:

$$\Delta(X) = \sum_{a\in\Sigma} \underbrace{\langle a|X|a\rangle}_{X_{a,a}} \cdot |a\rangle\langle a|, \quad \forall X \in L(\mathcal{X}).$$

The output of Δ is always a diagonal matrix.

Classical states and registers

A quantum state ρ on register X with space $\mathcal{X} = \mathbb{C}^\Sigma$ is classical if

$$\rho = \sum_{a\in\Sigma} p_a |a\rangle\langle a|.$$

Since $\rho \geq 0$ and $\mathrm{Tr}\,\rho = 1$, we have $p_a \geq 0$ and $\sum_{a\in\Sigma} p_a = 1$, so $\{p_a : a \in \Sigma\}$ is a probability distribution.
So classical states correspond precisely to probability distributions. Alternatively, $\rho \in D(\mathcal{X})$ is classical if $\Delta(\rho) = \rho$, i.e., the dephasing channel perfectly transmits classical information and it turns quantum information classical. A register X is classical if $(\Delta_X \otimes I_Y)(\rho_{XY}) = \rho_{XY}$ at all times, for any Y. In other words,

$$\rho_{XY} = \sum_{a\in\Sigma} p_a |a\rangle\langle a|_X \otimes (\rho_a)_Y,$$

for some probability distribution p_a and states ρ_a.
Such state ρ_{XY} is called classical-quantum.

Quantum information can thus be thought of as a non-commutative extension of classical information theory. This is precisely the sense in which many results in quantum information generalize their classical counterparts, i.e., from vectors (or diagonal matrices) to arbitrary matrices.

Quantum-to-classical channels

Def 2.35 A channel $\Phi \in C(\mathcal{X}, \mathcal{Y})$ is quantum-to-classical if $\Phi = \Delta \circ \Phi$ where $\Delta \in C(\mathcal{Y})$ is the completely dephasing channel.

Equivalently, $\Phi(\rho)$ is diagonal, for every $\rho \in D(\mathcal{X})$.

Measurements

(Also known as POVM.)

Def 2.34 A measurement is a function $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ such that $\sum_{a \in \Sigma} \mu(a) = I_{\mathcal{X}}$. We call Σ the set of measurement outcomes and $\mu(a)$ the measurement operator associated with outcome $a \in \Sigma$.

When a state $\rho \in D(\mathcal{X})$ on register X is measured, the following happens:

1. A random outcome $a \in \Sigma$ is selected with probability $p_a = \mathrm{Tr}[\mu(a)\rho]$.
2. The register X ceases to exist.

Example 2.41 The standard basis measurement in $\mathcal{X} = \mathbb{C}^\Sigma$ is $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ defined as $\mu(a) = |a\rangle\langle a|$. When measuring state $\rho \in D(\mathcal{X})$, we get outcome $a \in \Sigma$ with probability $p_a = \langle a|\rho|a\rangle = \rho_{a,a}$.

Example A measurement is called projective (or von Neumann) if $\mu(a)^2 = \mu(a)$, for all $a \in \Sigma$. I.e., each $\mu(a)$ is a projector (has eigenvalues 0/1). Say, $\Sigma = \{0,1\}$, $\mathcal{X} = \mathbb{C}^\Gamma$, $\Gamma = \{0,1,2,3\}$, $\mu(0) = |0\rangle\langle 0| + |2\rangle\langle 2|$, $\mu(1) = |1\rangle\langle 1| + |3\rangle\langle 3|$.

General measurements from projective ones

An isometry from $\mathcal{X}$ to $\mathcal{Z}$ is a map $A \in L(\mathcal{X}, \mathcal{Z})$ such that $A^*A = I_{\mathcal{X}}$. This means the columns of A are orthonormal:

$$\boxed{\quad A^* \quad} \cdot \boxed{A} = \boxed{\begin{smallmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{smallmatrix}}$$

In particular, this means $\dim \mathcal{Z} \geq \dim \mathcal{X}$.

Any such A can be competed to a unitary matrix by adding more orthonormal columns. We denote the set of all isometries from $\mathcal{X}$ to $\mathcal{Z}$ by $U(\mathcal{X}, \mathcal{Z})$.

Thm 2.42 (Naimark's) Let $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ be a measurement and let $\mathcal{Y} = \mathbb{C}^\Sigma$. Then there exists an isometry $A \in U(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ such that

$$\mu(a) = A^*(I_{\mathcal{X}} \otimes |a\rangle\langle a|)A.$$

Proof: Let $A = \sum_{a \in \Sigma} \sqrt{\mu(a)} \otimes |a\rangle$. Then

$$A^*A = \sum_{a,b \in \Sigma} \left(\sqrt{\mu(a)} \cdot \sqrt{\mu(b)}\right) \otimes \langle a|b\rangle = \sum_{a \in \Sigma} \mu(a) = I_{\mathcal{X}}. \qquad \square$$

Corollary 2.43 Let $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ be a measurement. Let $\mathcal{Y} = \mathbb{C}^\Sigma$ and $|u\rangle \in \mathcal{Y}$ be any unit vector. Then there exists a projective measurement $\nu : \Sigma \to \mathrm{Pos}(\mathcal{X}, \mathcal{Y})$ such that $\mathrm{Tr}[\nu(a) \cdot (X \otimes |u\rangle\langle u|)] = \mathrm{Tr}[\mu(a) \cdot X]$, for every $X \in L(\mathcal{X})$.

Proof: Let $A \in U(\mathcal{X}, \mathcal{X} \otimes \mathcal{Y})$ be the isometry from Naimark's theorem. Complete it to a unitary $U' \in U(\mathcal{X} \otimes \mathcal{Y})$ such that $U'(I_{\mathcal{X}} \otimes |0\rangle_{\mathcal{Y}}) = A$. Let $U = U' \cdot (I_{\mathcal{X}} \otimes V_{\mathcal{Y}})$ where $V_{\mathcal{Y}}|u\rangle_{\mathcal{Y}} = |0\rangle_{\mathcal{Y}}$ so that $U(I_{\mathcal{X}} \otimes |u\rangle_{\mathcal{Y}}) = A$. Let $\nu(a) = U^*(I_{\mathcal{X}} \otimes |a\rangle\langle a|_{\mathcal{Y}})U$. Then

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$$\mathrm{Tr}[\nu(a) \cdot (X \otimes |u\rangle\langle u|)] = \mathrm{Tr}[U^* (I \otimes |a\rangle\langle a|) \cdot U \cdot (I \otimes |u\rangle) \cdot (X \otimes \mathbb{1}) \cdot (I \otimes \langle u|)]$$

$$= \mathrm{Tr}\Big[\underbrace{(I \otimes \langle u|) \cdot U^*}_{A^*} \cdot (I \otimes |a\rangle\langle a|) \cdot \underbrace{U \cdot (I \otimes |u\rangle)}_{A} \cdot X\Big] = \mathrm{Tr}\Big[\underbrace{A^* \cdot (I \otimes |a\rangle\langle a|) \cdot A}_{\mu(a)} \cdot X\Big]. \qquad \square$$

Adjoint of a channel

The <u>adjoint</u> of a superoperator $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is $\Phi^* \in T(\mathcal{Y}, \mathcal{X})$ such that

$$\langle \Phi^*(Y), X \rangle = \langle Y, \Phi(X) \rangle,$$

for all $X \in L(\mathcal{X})$, $Y \in L(\mathcal{Y})$, where

$$\langle A, B \rangle = \mathrm{Tr}[A^* B] = \sum_{i,j} \overline{A}_{ij} B_{ij} = \langle \mathrm{vec}(A) | \mathrm{vec}(B) \rangle$$

is called the Hilbert-Schmidt inner product.

<u>Exercise</u>: Φ is positive if and only if Φ^* is (Prop. 2.18). Φ is trace-preserving if and only if Φ^* is <u>unital</u>, i.e., $\Phi^*(I_Y) = I_X$ (Thm. 2.26).

Measurements as channels

Quantum-to-classical channels $\Phi \in C(\mathcal{X}, \mathcal{Y})$ correspond precisely to measurements $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ on register X that store the outcome in a register Y with $\mathcal{Y} = \mathbb{C}^\Sigma$.

<u>Theorem 2.37</u>:

1. For every q-c channel $\Phi \in C(\mathcal{X}, \mathcal{Y})$ there exists a unique measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ such that

$$\Phi(X) = \sum_{a \in \Sigma} \mathrm{Tr}[\mu(a) X] \cdot |a\rangle\langle a| \quad \text{for all } X \in L(\mathcal{X}).$$

2. For every measurement $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$, the above Φ is a q-c channel.

<u>Proof</u>: 1. Since Φ is q-c, $\Phi(X) = \Delta(\Phi(X))$. Note that

$$\Delta(\Phi(X)) = \sum_{a\in\Sigma} \underbrace{\langle a|\Phi(X)|a\rangle}_{\mathrm{Tr}[|a\rangle\langle a|\cdot\Phi(X)]} \cdot |a\rangle\langle a| = \sum_{a\in\Sigma} \mathrm{Tr}\left[\Phi^*(|a\rangle\langle a|)\cdot X\right]\cdot |a\rangle\langle a|.$$

If we let $\mu(a) = \Phi^*(|a\rangle\langle a|)$ then $\mu(a) \in \mathrm{Pos}(\mathcal{X})$ since Φ^* is positive. Moreover, Φ^* is also unital so

$$\sum_{a\in\Sigma} \mu(a) = \sum_{a\in\Sigma} \Phi^*(|a\rangle\langle a|) = \Phi^*\Big(\sum_{a\in\Sigma} |a\rangle\langle a|\Big) = \Phi^*(I_\mathcal{Y}) = I_\mathcal{X}$$

and hence μ is indeed a measurement, and

$$\Phi(X) = \sum_{a\in\Sigma} \mathrm{Tr}\left[\mu(a)\cdot X\right]\cdot |a\rangle\langle a|.$$

It is not hard to show that such measurement is unique.

2. If $\mu : \Sigma \to \mathrm{Pos}(\mathcal{X})$ is a measurement then the super-operator $\Phi \in \mathrm{T}(\mathcal{X},\mathcal{Y})$ has Choi representation $J(\Phi) \in \mathrm{L}(\mathcal{Y}\otimes\mathcal{X})$,

$$\begin{aligned}
J(\Phi) &= \sum_{a,b\in\Sigma} \Phi(|a\rangle\langle b|)\otimes|a\rangle\langle b| \\
&= \sum_{a,b\in\Sigma}\Big(\sum_{c\in\Sigma} \mathrm{Tr}\left[\mu(c)\cdot|a\rangle\langle b|\right]\cdot|c\rangle\langle c|\Big)\otimes|a\rangle\langle b| \\
&= \sum_{c\in\Sigma} |c\rangle\langle c| \otimes \underbrace{\sum_{a,b\in\Sigma} \langle b|\mu(c)|a\rangle\cdot|a\rangle\langle b|}_{\mu(c)^T} \\
&= \sum_{c\in\Sigma} |c\rangle\langle c|\otimes\mu(c)^T \geqslant 0 \quad \text{since } \mu(c)\geqslant 0, \text{ for all } c\in\Sigma.
\end{aligned}$$

Moreover, $\mathrm{Tr}_\mathcal{Y}[J(\Phi)] = \sum_{c\in\Sigma} \underbrace{\mathrm{Tr}[|c\rangle\langle c|]}_{1}\otimes\mu(c)^T = \sum_{c\in\Sigma}\mu(c)^T = I_\mathcal{X}$,

so Φ is trace-preserving. By Corollary 2.27, Φ is a channel. It is evident that Φ is q-c from its definition (the output state is always diagonal).

$\square$

Partial measurements

Consider a system of two registers X and Y. Let $\mu : \Sigma \to \mathrm{Pos}(\mathcal{Y})$ be a measurement on the second register. How do we compute the probabilities of the measurement outcomes and the remaining state on X?

Let $\rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$ be the state of interest.

1. The probability of outcome $a \in \Sigma$ is

$$p_a = \mathrm{Tr}\left[\left(I_X \otimes \mu(a)_Y\right) \cdot \rho_{XY}\right].$$

2. The remaining state on register X corresponding to outcome $a \in \Sigma$ is

$$\rho_X(a) = \frac{1}{p_a} \mathrm{Tr}_Y\left[\left(I_X \otimes \mu(a)_Y\right) \cdot \rho_{XY}\right].$$

Note that $\mathrm{Tr}\, \rho_X(a) = 1$ and $\rho_X(a) \geq 0$.

Alternatively, let $\Phi \in C(\mathcal{Y}, \mathcal{Z})$, $\mathcal{Z} = \mathbb{C}^\Sigma$, be the q-c channel corresponding to measurement μ:

$$\Phi(Y) = \sum_{a \in \Sigma} \mathrm{Tr}\left[\mu(a) \cdot Y\right] \cdot |a\rangle\langle a|.$$

Let $\rho_{XY} \in D(\mathcal{X} \otimes \mathcal{Y})$. Then

$$(\mathcal{I}_X \otimes \Phi)(\rho_{XY}) = \sum_{a \in \Sigma} \underbrace{\left(\mathrm{Tr}_Y\left[\left(I_X \otimes \mu(a)_Y\right) \cdot \rho_{XY}\right]\right)_X}_{} \otimes |a\rangle\langle a|_Z$$

$$= \sum_{a \in \Sigma} \left(p_a \rho_X(a)\right) \otimes |a\rangle\langle a|_Z .$$

This is a block-diagonal state whose blocks contain $p_a \cdot \rho_X(a)$.

Non-destructive measurements

In the original definition of a measurement, the measured register disappears. Sometimes it is useful to have a post-measurement state around.

Let $\Phi \in C(\mathcal{X}, \mathcal{Y})$ whose Kraus representation is

$$\{M_a : a \in \Sigma\} \subset L(\mathcal{X}, \mathcal{Y}).$$

That is, $\Phi(X) = \sum_{a \in \Sigma} M_a X M_a^*$ where $\sum_{a \in \Sigma} M_a^* M_a = I_{\mathcal{X}}$.

Let $\Phi' \in C(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z})$, $\mathcal{Z} = \mathbb{C}^\Sigma$, be a channel with Kraus operators

$$\{M_a \otimes |a\rangle : a \in \Sigma\} \subset L(\mathcal{X}, \mathcal{Y} \otimes \mathcal{Z}).$$

This is indeed a channel since

$$\sum_{a \in \Sigma} (M_a \otimes |a\rangle)^* (M_a \otimes |a\rangle) = \sum_{a \in \Sigma} M_a^* M_a \otimes \langle a|a\rangle = I_{\mathcal{X}}.$$

If $\rho_X \in D(\mathcal{X})$ then $\Phi'(\rho_X) = \sum_{a \in \Sigma} \underbrace{M_a \rho_X M_a^*}_{p(a)\, \sigma_Y(a)} \otimes |a\rangle\langle a|_Z \in D(\mathcal{Y}, \mathcal{Z})$

is a q-c state whose first register contains post-measurement states $\sigma_Y(a)$ and probabilities $p(a)$.

When a <u>non-destructive measurement</u> corresponding to operators $\{M_a : a \in \Sigma\} \subset L(\mathcal{X}, \mathcal{Y})$ is performed on $\rho_X \in D(\mathcal{X})$:

1. Outcome $a \in \Sigma$ is obtained with probability
$$p(a) = \mathrm{Tr}\left[M_a \rho_X M_a^*\right].$$
2. The corresponding post-measurement state is
$$\sigma_Y(a) = \frac{1}{p(a)} M_a \rho_X M_a^*.$$

Instruments

A somewhat more general notion of a non-destructive measurement exsist where several measurement outcomes can be grouped together. This is similar to how a general projective measurement can be thought of as a standard basis measurement where several outcomes are combined.

Let $\{\Phi_a : a \in \Sigma\} \subset CP(\mathcal{X}, \mathcal{Y})$ be a collection of completely positive maps (which may not be trace-preserving) such that $\Phi = \sum_{a \in \Sigma} \Phi_a$ is trace-preserving and hence a channel. Such Φ_a are known as <u>subchannels</u> and their collection is known as an <u>instrument</u>.

When an instrument is applied to $\rho \in D(\mathcal{X})$:

1. An outcome $a \in \Sigma$ is selected at random with probability
$$p(a) = \mathrm{Tr}\left[\Phi_a(\rho)\right].$$

2. The corresponding post-measurement state is
$$\rho(a) = \frac{1}{p(a)} \Phi_a(\rho).$$

In the special case when each Φ_a has just one Kraus operator, i.e., $\Phi_a(\rho) = M_a \rho M_a$, for some $M_a \in L(\mathcal{X}, \mathcal{Y})$, we recover a regular non-destructive measurement.

Note that any instrument can be implemented by a channel Φ' similar to the one defined for non-destructive measurements.