Quantum Information Theory, Spring 2019

Problem Set 1 **due February 11, 2019**

1. *(4 points)* **A formula for the trace distance between pure states:** Consider two pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$ on a Hilbert space $\mathcal{X}$. Show that

$$\frac{1}{2}\|\rho - \sigma\|_1 = \sqrt{1 - |\langle\psi|\phi\rangle|^2}.$$

Hint: Can you reduce to the situation where $\mathcal{X} = \mathbb{C}^2$?

2. *(4 points)* **Uncertainty relation:** Given a measurement $\mu \colon \{0,1\} \to \mathrm{Pos}(\mathcal{X})$ with two outcomes and a state $\rho \in D(\mathcal{X})$, define the *bias* by

$$\beta(\rho) = \big|\mathrm{tr}[\mu(0)\rho] - \mathrm{tr}[\mu(1)\rho]\big|.$$

Note that $\beta = 1$ iff the outcome is deterministic and $\beta = 0$ iff both outcomes are equally likely. In class, we discussed how to measure a qubit in the standard basis and in the Hadamard basis. Let β_{std} and β_{Had} denote the bias for these two measurements.

(a) Show that, for every qubit state ρ,

$$\beta_{\mathrm{std}}(\rho) = |\mathrm{tr}[Z\rho]| \quad \text{and} \quad \beta_{\mathrm{Had}}(\rho) = |\mathrm{tr}[X\rho]|,$$

where X and Z are two of the three Pauli matrices defined in class.

(b) Show that, for every qubit state ρ,

$$\beta_{\mathrm{std}}(\rho) + \beta_{\mathrm{Had}}(\rho) \leq \sqrt{2}.$$

Why is it appropriate to call this an *uncertainty relation*?

(c) Find a state ρ for which the uncertainty relation is saturated (i.e., an equality).

3. *(4 points)* **No cloning:** In this problem, you will show that it is not possible to perfectly clone an unknown state – even if we restrict to classical or to pure states. Let $\mathcal{X} = \mathbb{C}^\Sigma$ be a qubit, i.e., $\Sigma = \{0,1\}$. We say that a channel $\Phi \in C(\mathcal{X}, \mathcal{X} \otimes \mathcal{X})$ *clones* a state $\rho \in D(\mathbb{C}^2)$ if $\Phi[\rho] = \rho \otimes \rho$.

(a) Show that there exists no channel that clones all classical states ρ.

(b) Show that there exists no channel that clones all pure states ρ.

(c) Which states are both pure and classical? Find a channel Φ that clones all of them.

Hint: For (a) and (b), use linearity and the cloning property to arrive at a contradiction.

4. *(4 points)* **Practice:** The file `pset1.txt` on the course homepage contains the matrix representation (row by row) of a state ρ on $\mathcal{X} = \mathbb{C}^\Sigma$, where $\Sigma = \{0, \dots, d-1\}$.

(a) What is the dimension d of the quantum system that ρ is a state on?

(b) Compute the largest eigenvalue of ρ. Is ρ a pure state?

(c) Let σ be the state obtained by applying the depolarizing channel $\Phi[\rho] = 0.3\rho + 0.7\frac{I}{d}$ to ρ. What is the largest eigenvalue of σ?

(d) Compute the trace distance $\|\sigma - \frac{I}{d}\|_1$ between σ and the maximally mixed state on $\mathcal{X}$.

(e) Imagine measuring σ in the standard basis. Which outcome $x \in \Sigma$ has *smallest* probability?

Hint: On the course homepage you can find instructions for loading the matrix.