Quantum Information Theory, Spring 2019

Problem Set 10 **due April 15, 2019**

1. *(2 points)* **Discriminating Bell states by LOCC:** Recall that the Bell states are given by

$$|\Phi^{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \qquad |\Phi^{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\Phi^{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \qquad |\Phi^{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Assume that Alice holds the first qubit of a Bell state and Bob holds the second qubit.

(a) Find an LOCC protocol that can perfectly discriminate between $|\Phi^{00}\rangle$ and $|\Phi^{01}\rangle$.
(b) Find an LOCC protocol that can perfectly discriminate between $|\Phi^{00}\rangle$ and $|\Phi^{10}\rangle$.

2. *(6 points)* **One-way LOCC struggle:** Let $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$. Consider a two-qubit system where Alice holds the first qubit and Bob holds the second qubit. These two qubits are initialized in one of the following four states:

$$|\Psi_1\rangle = |0\rangle \otimes |0\rangle,$$
$$|\Psi_2\rangle = |0\rangle \otimes |1\rangle,$$
$$|\Psi_3\rangle = |1\rangle \otimes |+\rangle,$$
$$|\Psi_4\rangle = |1\rangle \otimes |-\rangle.$$

(a) Show that if μ is a separable measurement, and μ perfectly distinguishes an orthonormal basis, then this basis must consist of product states.
(b) Write down a measurement that perfectly distinguishes the above four states and show that it is separable.
(c) Find a one-way LOCC measurement from Alice to Bob that perfectly determines which of the four states they share.
(d) Show that there is no one-way LOCC measurement from Bob to Alice that can perfectly determine which of the four states they share.
Hint: Show that the choice of measurement for Alice can not depend on the outcome of Bob if she wants to perfectly distinguish the remaining states on her qubit.

3. *(4 points)* **Operations on PPT states:** Suppose that Alice and Bob share a PPT (positive partial transpose) state ρ_{AB}.

(a) Show that if they apply a separable channel Ξ, the resulting state $\Xi(\rho_{\mathsf{AB}})$ is again PPT.
(b) Show that they cannot get a maximally entangled state

$$|\Phi^+_{\mathsf{AB}}\rangle = \frac{1}{\sqrt{|\Sigma|}} \sum_{a \in \Sigma} |a\rangle_{\mathsf{A}} \otimes |a\rangle_{\mathsf{B}}$$

with any $|\Sigma| > 1$ by applying an LOCC operation on ρ_{AB}.

4. *(4 points)* **Practice:** Alice and Bob share a *real* two-qubit state which is either $|\Psi_1\rangle$ or $|\Psi_2\rangle$, which are provided in files `psi1.txt` and `psi2.txt`, respectively, and are promised to be orthogonal. The goal of Alice and Bob is to perfectly discriminate these two states by a one-way LOCC protocol from Alice to Bob. Surprisingly, this can always be done! Moreover, all involved measurements are orthonormal measurements in basis $B(\alpha) = \{|v(\alpha)\rangle, |v(\alpha + \pi/2)\rangle\}$, for some angle $\alpha \in [0, \pi)$, where

$$|v(\alpha)\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}. \tag{1}$$

   (a) Consider the function $f(\alpha) = \langle\Psi_1|\big(|v(\alpha)\rangle\langle v(\alpha)| \otimes I\big)|\Psi_2\rangle$. Plot this function $f(\alpha)$ and determine an angle α that Alice should use in her measurement.
   Hint: The post-measurement states on Bob's side should be orthogonal.

   (b) Alice sends the binary outcome of her measurement to Bob who then measures in a basis $B(\beta)$, where the angle β depends on the outcome he received from Alice. For each outcome of Alice's measurement, determine what angle Bob should use in his measurement.