Quantum Information Theory, Spring 2019

Problem Set 12 **due May 6, 2019**

1. *(4 points)* **Fidelity inequality:** Let $\mathcal{X}$ be a complex Euclidean space with $\dim(\mathcal{X}) \geq 2$.

   (a) Let $|u_1\rangle, |u_2\rangle, |v\rangle \in \mathcal{X}$ be arbitrary pure quantum states. Show that

   $$|\langle u_1|v\rangle|^2 + |\langle u_2|v\rangle|^2 \leq 1 + |\langle u_1|u_2\rangle|.$$

   Hint: Upper bound the left-hand side by the largest eigenvalue of some rank-2 matrix. Compute this eigenvalue to get the right-hand side.

   (b) Let $\rho_1, \rho_2, \sigma \in \mathrm{D}(\mathcal{X})$ be arbitrary states. Show that

   $$\mathrm{F}(\rho_1, \sigma)^2 + F(\rho_2, \sigma)^2 \leq 1 + \mathrm{F}(\rho_1, \rho_2).$$

2. *(4 points)* **Entanglement cost using compression and teleportation:**
   In this exercise you will give an alternative proof for the fact that the entanglement cost is at most the entanglement entropy for a pure state. Let $|\psi\rangle_{\mathsf{XY}} \in \mathcal{S}(\mathcal{X} \otimes \mathcal{Y})$ be a pure state.

   (a) Let ρ_{X} and ρ_{Y} be the reduced density matrices and let $\alpha > \mathrm{H}(\rho_{\mathsf{X}}) = \mathrm{H}(\rho_{\mathsf{Y}})$. Show, using compression, that for all $\delta > 0$ there exists an LOCC protocol for all but finitely many n which converts $\lfloor \alpha n \rfloor$ Bell pairs into a state $\tilde{\psi}_n$ with $F(\psi^{\otimes n}, \tilde{\psi}_n) > 1 - \delta$.
   Hint: Use teleportation!

   (b) Use (a) to show that for every pure state $\mathrm{E_C}(\mathsf{X} : \mathsf{Y}) \leq \mathrm{H}(\mathsf{X})$.

3. *(3 points)* **Entanglement rank and the fidelity with the maximally entangled state:**
   Let $\Sigma = \{1, \dots, n\}$ and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^\Sigma$ be two complex Euclidean spaces of dimension n. Let $\tau_n \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ be the canonical maximally entangled state, i.e., $\tau_n = |\tau_n\rangle\langle\tau_n|$ where

   $$|\tau_n\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} |i\rangle \otimes |i\rangle.$$

   Show that, for any state $\rho \in \mathrm{D}(\mathcal{X} \otimes \mathcal{Y})$ of entanglement rank r, $\mathrm{F}(\tau_n, \rho)^2 \leq r/n$.

4. *(5 points)* **Entanglement distillation:** For any $p \in [0, 1]$, let

   $$|\tau(p)\rangle_{\mathsf{AB}} = \sqrt{p}\, |0\rangle \otimes |0\rangle + \sqrt{1-p}\, |1\rangle \otimes |1\rangle$$

   be a two-qubit state shared between Alice and Bob, and let $|\tau\rangle_{\mathsf{AB}} = |\tau(1/2)\rangle_{\mathsf{AB}}$ denote the *maximally entangled state*. In the following, we always assume that Alice and Bob are given $|\tau(p)\rangle_{\mathsf{AB}}^{\otimes n}$, for some $n \geq 1$ and $0 \leq p \leq 1/2$, as their input.

   (a) Using LOCC, they would like to distill one perfect copy of $|\tau(q)\rangle_{\mathsf{AB}}$ that is as close as possible to the maximally entangled state $|\tau\rangle_{\mathsf{AB}}$. Derive a formula for the largest possible $q \leq 1/2$ they can get and express it as an explicit function of p and n. What is the maximal fidelity their output state $|\tau(q)\rangle_{\mathsf{AB}}$ can have with the desired target state $|\tau\rangle_{\mathsf{AB}}$, i.e., what is

   $$\max_q \mathrm{F}\big(|\tau(q)\rangle, |\tau\rangle\big),$$

   where the maximum is over all possible achievable q?

(b) Take $p = 0.1$ and plot this as a function of n. Determine the smallest n such that $|\tau(p)\rangle_{\mathsf{AB}}^{\otimes n}$ can be perfectly transformed into $|\tau\rangle_{\mathsf{AB}}$ by LOCC.

(c) This time Alice and Bob want to extract more entanglement. That is, they want to perfectly obtain the state

$$|\tau_d\rangle_{\mathsf{A}'\mathsf{B}'} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle_{\mathsf{A}'} |i\rangle_{\mathsf{B}'},$$

for some $d \geq 2$. Derive a formula for the largest possible d as a function of p and n.

(d) Take $p = 0.1$ and plot this as a function of n. Determine the smallest n such that $|\tau(p)\rangle_{\mathsf{AB}}^{\otimes n}$ can be perfectly transformed into three copies of $|\tau\rangle_{\mathsf{AB}}$ by LOCC.

(e) Take $p = 0.2$ and $n = 2$. What is the largest possible fidelity their output state can have with $|\tau_3\rangle_{\mathsf{AB}}$?