Quantum Information Theory, Spring 2019

Problem Set 14 **due May 20, 2019**

1. *(4 points)* **Fidelity and measurements:** Recall that the fidelity between two states $\rho, \sigma \in \mathrm{D}(\mathcal{X})$ is defined as $F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$. We can similarly define the *fidelity* (or *Bhattacharyya coefficient*) between two probability distributions $p, q \in \mathcal{P}(\Sigma)$ by $F(p, q) = \sum_{x \in \Sigma} \sqrt{p(x)}\sqrt{q(x)}$.

   (a) Show that $F(\rho, \sigma) \leq F(p, q)$ for every measurement $\mu \colon \Sigma \to \mathrm{Pos}(\mathcal{X})$, where p and q denote the probability distribution of outcomes when measuring on ρ and σ, respectively.

   (b) Show that there exists a measurement such that equality holds in part (a).

   Hint: You may assume that ρ is invertible. Consider the measurement in the eigenbasis of the operator $M := \rho^{-1/2}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}\rho^{-1/2}$, which satisfies $M\rho M = \sigma$. See also Problem 4.

2. *(4 points)* **Fuchs-van de Graaf inequalities:** The fidelity and trace distance are related by

$$1 - \frac{1}{2}\|\rho - \sigma\|_1 \leq F(\rho, \sigma) \leq \sqrt{1 - \frac{1}{4}\|\rho - \sigma\|_1^2}$$

   known as the *Fuchs-van de Graaf inequalities.* You proved the second inequality in Exercise 5.2. You will now prove the first inequality.

   (a) Show that $|a - b| \geq \left(\sqrt{a} - \sqrt{b}\right)^2$ for all $a, b \in [0, 1]$.

   (b) Conclude that $1 - \frac{1}{2}\|\rho - \sigma\|_1 \leq F(\rho, \sigma)$ holds for any pair of states $\rho, \sigma \in \mathrm{D}(\mathcal{X})$.

   Hint: Use Problem 1 to reduce the claim to probability distributions.

3. *(6 points)* **Proof of the decoupling theorem:** In this problem you will prove the *decoupling theorem* that we used in class. It states that, for every state $\rho_{AR} \in \mathrm{D}(\mathcal{A} \otimes \mathcal{R})$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$,

$$\int \left\| \mathrm{Tr}_{A_1}\left[(U_A^\dagger \otimes I_R)\rho_{AR}(U_A \otimes I_R)\right] - \frac{I_{A_2}}{d_{A_2}} \otimes \rho_R \right\|_1^2 dU_A \leq \frac{d_A d_R}{d_{A_1}^2} \mathrm{Tr}[\rho_{AR}^2]. \qquad (1)$$

   Here, dU_A denotes the Haar measure on $\mathrm{U}(\mathcal{A})$ from last week's homework, and $d_A = \dim \mathcal{A}$ etc.

   (a) Show that $\int U_A^{\otimes 2}\left(I_{A_1A_1} \otimes F_{A_2A_2}\right) U_A^{\dagger, \otimes 2}\, dU_A = \alpha I_{AA} + \beta F_{AA}$ for constants $\alpha \leq \frac{1}{d_{A_2}}$, $\beta \leq \frac{1}{d_{A_1}}$.

   (b) Deduce that $\int \mathrm{Tr}\Big[\mathrm{Tr}_{A_1}\left[(U_A^\dagger \otimes I_R)\rho_{AR}(U_A \otimes I_R)\right]^2\Big]\, dU_A = \alpha\,\mathrm{Tr}[\rho_R^2] + \beta\,\mathrm{Tr}[\rho_{AR}^2]$.

   (c) Conclude the proof of the decoupling theorem, i.e., show that Eq. (1) holds.

   Hint: For (a) and (b), you can build on results from last week's homework. For (c), start with the bound $\|M\|_1^2 \leq d_X \|M\|_2^2 = d_X \,\mathrm{Tr}[M^\dagger M]$ which holds for any operator $M \in L(\mathcal{X})$.

4. *(2 points)* **Practice:** Let Alice, Bob, and a reference system share many copies of a pure state Ψ_{ABR}. Using quantum state merging, by sending qubits at rate $I(A:R)/2$, Alice can approximately transfer her part of the state to Bob and simultaneously 'distill' Bell pairs $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ at rate $I(A:B)/2$ between herself and Bob. Show that, for the state $|\Psi_{ABR}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, this can in fact be done *exactly* by using only *two copies.*

   Hint: Let Alice apply the unitary $U = \mathrm{CNOT}(H \otimes I)$ to her two qubits and send her second qubit to Bob. You do not need to find Bob's decoding isometry explicitly.