Quantum Information Theory, Spring 2019

Problem Set 2 **due February 18, 2019**

1. *(4 points)* **Nayak's bound:** Alice wants to communicate m bits to Bob by sending n qubits. She chooses one state $\rho_x \in D(\mathcal{X})$, $\mathcal{X} = (\mathbb{C}^2)^{\otimes n}$, for each possible message $x \in \{0,1\}^m$ that she may want to send, and Bob chooses a measurement $\mu\colon \{0,1\}^m \to \mathrm{Pos}(\mathcal{X})$ that he will use to decode the message.

   (a) Write down a formula for the probability that Bob successfully decodes the message if the message is drawn according to an arbitrary probability distribution $p(x)$ on $\{0,1\}^m$.
   (b) Show that the probability that Bob successfully decodes the bitstring is at most 2^{n-m} if the message is drawn uniformly at random.

2. *(4 points)* **Extensions of pure states:** Let X, Y, Z be arbitrary quantum systems with Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, respectively.

   (a) Show that $\mathrm{tr}_Z[A \otimes N] = A \otimes \mathrm{tr}_Z[N]$ for all $A \in L(\mathcal{X})$ and $N \in L(\mathcal{Y} \otimes \mathcal{Z})$.
   (b) Show that $\rho = \rho_X \otimes \rho_Y$ for every state $\rho \in D(\mathcal{X} \otimes \mathcal{Y})$ such that ρ_X is pure.
   (c) Show that $\rho_{XZ} = \rho_X \otimes \rho_Z$ for every state $\rho \in D(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$ such that ρ_{XY} is pure.
   (d) Show that $\rho = \rho_X \otimes \rho_Y \otimes \rho_Z$ for every $\rho \in D(\mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{Z})$ such that ρ_{XY} and ρ_{XZ} are pure.

   Hint: In class we proved (b) when ρ is pure. Can you use a purification to reduce to this case?

3. *(4 points)* **Properties of the fidelity:** Use Uhlmann's theorem to prove the following two properties of the fidelity. As before, X, Y denote arbitrary systems with Hilbert spaces $\mathcal{X}$, $\mathcal{Y}$.

   (a) *Monotonicity:* $F(\rho, \sigma) \leq F(\rho_X, \sigma_X)$ for any two states $\rho, \sigma \in D(\mathcal{X} \otimes \mathcal{Y})$.
   (b) *Joint concavity:* $F(\sum_{i\in I} p_i \rho_i, \sum_{i \in I} p_i \sigma_i) \geq \sum_{i \in I} p_i F(\rho_i, \sigma_i)$, where $(p_i)_{i\in I}$ is an arbitrary finite probability distribution and where $(\rho_i)_{i \in I}$, $(\sigma_i)_{i\in I}$ are families of states in $D(\mathcal{X})$.

   Hint: Can you prove the inequalities by finding suitable purifications?

4. *(4 points)* **Practice:** The file `pset2.txt` contains the entries of a complex 2×2-matrix A (row by row). The corresponding vector $|\Psi\rangle = \sum_{x,y} A_{x,y} |x\rangle \otimes |y\rangle \in \mathcal{X} \otimes \mathcal{Y}$, where $\mathcal{X} = \mathcal{Y} = \mathbb{C}^2$, has norm one and hence determines a pure two-qubit state $\rho = |\Psi\rangle\langle\Psi|$.

   (a) Compute the Schmidt coefficients of $|\Psi\rangle$.
   (b) Compute the reduced states ρ_X and ρ_Y.
   (c) Compute the trace distance $\|\rho_X - \rho_Y\|_1$.
   (d) Compute the fidelity $F(\rho_X, \rho_Y)$.

   Hint: On the course homepage you can find instructions for loading the complex matrix.

Notation: *In Problems 2–4, just like in class, ρ_X refers to the reduced state of ρ obtained by taking the partial trace over all systems other than X (and likewise for ρ_Y etc.).*