Quantum Information Theory, Spring 2019

Problem Set 3 due February 25, 2019

1. (4 points) **Vectorization:** Recall that if $\mathcal{X} = \mathbb{C}^\Sigma$ and $\mathcal{Y} = \mathbb{C}^\Gamma$ then $\text{vec} : L(\mathcal{Y}, \mathcal{X}) \rightarrow \mathcal{X} \otimes \mathcal{Y}$ is defined as $\text{vec}(|i\rangle \langle j|) = |i\rangle \otimes |j\rangle$, for all $i \in \Sigma$ and $j \in \Gamma$, and then extended by linearity.

(a) Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}, \mathcal{Y}'$ be complex Euclidean spaces and let $A \in L(\mathcal{X}, \mathcal{X}')$, $B \in L(\mathcal{Y}, \mathcal{Y}')$, and $X \in L(\mathcal{Y}, \mathcal{X})$. Show that $(A \otimes B) \text{vec}(X) = \text{vec}(AXB^T)$.

(b) Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\sigma \in D(\mathcal{X})$. Recall that the standard purification of $\sigma$ is given by $|\psi\rangle = (\sqrt{\sigma} \otimes I_X) \cdot \sum_{x \in \Sigma} |x\rangle \otimes |x\rangle$.

Show that $|\psi\rangle = \text{vec}(\sqrt{\sigma})$.

2. (2 points) **Quantum channels:** Show that the following maps $\Phi$ are quantum channels by directly verifying that they are trace-preserving and completely positive.

(a) (Mixed unitary): Let $(p_1, \ldots, p_n)$ be a probability distribution, let $U_1, \ldots, U_n \in U(\mathcal{X})$ be a set of unitary matrices, and let $\Phi \in T(\mathcal{X})$ be defined as follows:

$$\Phi(X) = \sum_{i=1}^{n} p_i U_i X U_i^*.$$

(b) (State preparation): Let $\sigma \in D(\mathcal{X})$ and let $\Phi \in T(\mathcal{X})$ be defined as follows:

$$\Phi(X) = \text{Tr}[X] \sigma. \quad (1)$$

3. (2 points) **Kraus vs Choi:** Recall that the Kraus representation of a superoperator $\Phi \in T(\mathcal{X}, \mathcal{Y})$ is given by

$$\Phi(X) = \sum_{a \in \Gamma} A_a X B_a^*,$$

for some operators $\{A_a : a \in \Gamma\}, \{B_a : a \in \Gamma\} \subset L(\mathcal{X}, \mathcal{Y})$. Show that the Choi representation of the same superoperator is given by

$$J(\Phi) = \sum_{a \in \Gamma} \text{vec}(A_a) \text{vec}(B_a)^*.$$

4. (4 points) **Kraus operators:** Derive a Kraus representation for the following quantum channels:

(a) (Discarding the input): Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\Phi \in C(\mathcal{X}, \mathcal{C})$ be the quantum channel that discards the input. That is, for all $X \in L(\mathcal{X})$,

$$\Phi(X) = \text{Tr}[X].$$

(b) (State preparation): Let $\mathcal{X} = \mathbb{C}^\Sigma$ and $\Phi \in C(\mathcal{C}, \mathcal{X})$ be the quantum channel that discards the input and replaces it by some fixed state $\sigma \in D(\mathcal{X})$, see Eq. (1).
5. (4 points) Practice: The files pset3-A1.txt, pset3-A2.txt, pset3-A3.txt contain the entries of three $5 \times 5$ matrices $A_1, A_2, A_3 \in L(X)$ where $X = \mathbb{C}^5$. These matrices define a superoperator $\Phi \in T(X)$ that acts as

$$\Phi(X) = \sum_{i=1}^{3} A_i X A_i^*.$$

(a) Compute the eigenvalues of the Choi matrix $J(\Phi)$.
(b) Verify that $\Phi$ is a quantum channel (explain what you did).

*Hint: On the course homepage you can find instructions for loading a complex matrix.*