Quantum Information Theory, Spring 2019

Problem Set 6  
due March 18, 2019

1. (4 points) **Compression and correlations:** In this problem, you will show that the definition of an \((n, R, δ)\)-quantum code is more stringent than demanding that the average fidelity is high for any particular source. Consider a qubit source that emits states \(\rho_0 = |0⟩⟨0|\) and \(\rho_1 = |1⟩⟨1|\) with 50% probability each. Let \(ρ ∈ D(\mathcal{X})\) denote the average output state of the source (\(\mathcal{X} = \mathbb{C}^2\)). Find channels \(\mathcal{E}_n \in C(\mathcal{X}^⊗n, (\mathbb{C}^2)^⊗n)\), \(\mathcal{D}_n \in C((\mathbb{C}^2)^⊗n, \mathcal{X}^⊗n)\) such that

\[
\sum_{x_1, \ldots, x_n} 2^{-n} F(\mathcal{D}_n[\mathcal{E}_n|\rho_{x_1} \otimes \cdots \otimes \rho_{x_n}], \rho_{x_1} \otimes \cdots \otimes \rho_{x_n}) = 1,
\]

but \(F(\mathcal{D}_n\mathcal{E}_n, ρ^⊗n) → 0\) as \(n → ∞\).

**Hint:** Choose both \(\mathcal{E}_n\) and \(\mathcal{D}_n\) to be measure-and-prepare channels.

2. (4 points) **Montonicity properties:** We already know that the trace distance and fidelity are monotone with respect to tracing out subsystems. In this problem, you will extend these properties to arbitrary quantum channels \(Φ ∈ C(\mathcal{X}, \mathcal{Y})\). Thus, show that:

(a) \(∥Φ[A]∥_1 ≤ ∥A∥_1\) for all \(A ∈ L(\mathcal{X})\), and hence \(∥Φ[ρ] − Φ[σ]∥_1 ≤ ∥ρ − σ∥_1\) for all \(ρ, σ ∈ D(\mathcal{X})\).

(b) \(F(Φ[ρ], Φ[σ]) ≥ F(ρ, σ)\) for all \(ρ, σ ∈ D(\mathcal{X})\).

**Hint:** First prove that \(||VAV^*||_1 = ||A||_1\) for every \(A ∈ L(\mathcal{X})\) and isometry \(V ∈ L(\mathcal{X}, \mathcal{W})\).

3. (4 points) **Subadditivity:** Use Schumacher’s theorem to prove the following inequality, which is known as the **subadditivity property** of the quantum entropy:

\[H(ρ_X) + H(ρ_Y) ≥ H(ρ)\]

for every \(ρ ∈ D(\mathcal{X} ⊗ \mathcal{Y})\) with reduced states \(ρ_X = \text{Tr}_Y[ρ]\) and \(ρ_Y = \text{Tr}_X[ρ]\).

**Hint:** You are allowed to use the following ‘triangle inequality’ for the fidelity (without proof): For any three states \(α, β, γ ∈ D(\mathcal{H})\), if \(F(α, β) ≥ 1 − δ\) and \(F(β, γ) ≥ 1 − δ\) then \(F(α, γ) ≥ 1 − 4δ\).

4. (4 points) **Practice:** In this problem, you can explore the behavior of the typical subspaces. Consider the state \(ρ = \frac{1}{2} |0⟩⟨0| + \frac{1}{2} |+⟩⟨+|\), where \(|+⟩ = \frac{1}{\sqrt{2}}(|0⟩ + |1⟩)\).

(a) Compute the largest eigenvalue \(p\) of \(ρ\) and the quantum entropy \(H(ρ)\).

(b) Plot the following functions for \(n = 100\) and \(n = 1000\):

\[d(k) = \binom{n}{k}, \quad r(k) = \frac{1}{n} \log \binom{n}{k}, \quad q(k) = \binom{n}{k} p^k (1 − p)^{n−k}\]

for \(k ∈ \{0, 1, \ldots, n\}\).

(c) Plot the following functions for \(ε = 0.1\) and \(ε = 0.01\):

\[r(n) = \frac{1}{n} \log \dim S_{n,ε}, \quad p(n) = \text{Tr} [\Pi_{n,ε} ρ^⊗n],\]

for \(n ∈ \{1, \ldots, 1000\}\), where \(Π_{n,ε}\) denotes the orthogonal projection onto the typical subspace \(S_{n,ε}\) of \(ρ\).