Quantum Information Theory, Spring 2019

Problem Set 9 **due April 8, 2019**

1. *(4 points)* **Teleportation and entanglement swapping:**

   (a) Let $|\psi\rangle \in \mathbb{C}^2$ be an arbitrary pure single-qubit state. Verify the teleportation identity

   $$|\psi\rangle \otimes |\Phi^{00}\rangle = \frac{1}{2} \sum_{z,x \in \{0,1\}} |\Phi^{zx}\rangle \otimes X^x Z^z |\psi\rangle,$$

   where $|\Phi^{zx}\rangle$ are the four Bell states and X and Z are the Pauli matrices.
   Hint: Some identities from the exercise class might be handy.

   (b) Consider the following generalization of the teleportation identity involving three friends Zoey, Alice, and Bob who share four qubits: Zoey has qubit 1, Alice has qubits 2 and 3, while Bob has qubit 4. Their joint state is

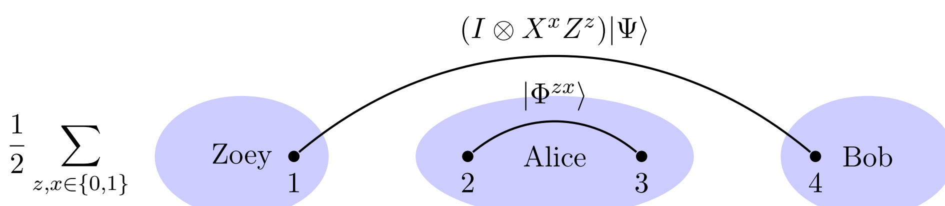
   $$|\Psi\rangle_{12} \otimes |\Phi^{00}\rangle_{34}$$

   where $|\Psi\rangle_{12} \in \mathcal{S}(\mathbb{C}^4)$ is an arbitrary pure two-qubit state. This can be depicted as follows:

   Show that

   $$|\Psi\rangle_{12} \otimes |\Phi^{00}\rangle_{34} = \frac{1}{2} \sum_{z,x \in \{0,1\}} |\Phi^{zx}\rangle_{23} \otimes (I \otimes X^x Z^z)|\Psi\rangle_{14}, \tag{1}$$

   where the subscripts indicate the qubits on which these states live. Note that pictorially the right-hand side looks as follows:



   (c) Using Eq. (1), explain what happens if Alice and Bob perform the usual teleportation protocol. More specifically, what are the probabilities of outcomes if Alice measures her two qubits in the Bell basis? For each outcome, what is the corresponding post-measurement state for Zoey and Bob? What Pauli correction operation should Bob apply at the end of the protocol?

   (d) What could be a potential application of the protocol performed by Zoey, Alice, and Bob?

2. **Maximally entangled state tricks:** Let $\mathcal{X} = \mathcal{Y} = \mathbb{C}^\Sigma$ and

   $$|\Phi^+\rangle = \frac{1}{\sqrt{|\Sigma|}} \sum_{j \in \Sigma} |j\rangle \otimes |j\rangle.$$

(a) Show that, for any $A \in \mathrm{L}(\mathcal{X})$,

$$(A \otimes I)|\Phi^+\rangle = (I \otimes A^{\mathsf{T}})|\Phi^+\rangle.$$

(b) Show that for $A, B \in \mathrm{L}(\mathcal{X})$

$$\mathrm{Tr}(A^* B) = |\Sigma| \langle \Phi^+ | \overline{A} \otimes B | \Phi^+ \rangle$$

(c) Suppose $\Psi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{X})$ is positive. Show that Ψ is completely positive if and only if $(\Psi \otimes \mathcal{I}_{\mathsf{X}})(|\Phi^+\rangle\langle\Phi^+|)$ is positive.

3. *(4 points)* **Partial transpose test:** Let Σ be an alphabet, $n = |\Sigma|$, and $\mathcal{X} = \mathcal{Y} = \mathbb{C}^{\Sigma}$. Let

$$|\Phi^+\rangle_{\mathsf{XY}} = \frac{1}{\sqrt{n}} \sum_{a \in \Sigma} |a\rangle_{\mathsf{X}} \otimes |a\rangle_{\mathsf{Y}}$$

be an n-dimensional *maximally entangled state* on registers XY. Let $\lambda \in [0, 1]$ and define $\Gamma_0, \Gamma_1, \rho(\lambda) \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$ as follows:

$$\Gamma_0 = |\Phi^+\rangle\langle\Phi^+|, \qquad \Gamma_1 = \frac{I \otimes I - \Gamma_0}{n^2 - 1}, \qquad \rho(\lambda) = \lambda\Gamma_0 + (1 - \lambda)\Gamma_1.$$

(a) Show that $\rho(\lambda)$ is a quantum state when $\lambda \in [0, 1]$.
(b) Let $\mathrm{T} \in \mathrm{T}(\mathcal{X})$ be the *transpose* map defined as $\mathrm{T}(X) = X^{\mathsf{T}}$, for all $X \in \mathrm{L}(\mathcal{X})$. Show that

$$(\mathrm{T}_{\mathsf{X}} \otimes \mathcal{I}_{\mathsf{Y}})(\Gamma_0) = \frac{1}{n} W,$$

where $W \in \mathrm{L}(\mathcal{X} \otimes \mathcal{Y})$ is the *swap operator* defined as

$$W(|x\rangle \otimes |y\rangle) = |y\rangle \otimes |x\rangle,$$

for all $x, y \in \Sigma$.
(c) Compute $(\mathrm{T}_{\mathsf{X}} \otimes \mathcal{I}_{\mathsf{Y}})(\rho(\lambda))$.
(d) For what range of λ can we say that $\rho(\lambda)$ is entangled?

4. *(4 points)* **Practice** In this problem, you will play around with the partial transpose test (part a) and the entanglement entropy (part b).

(a) In the files `A.txt`, `B.txt` and `C.txt`, you will find density matrices of the following dimensions:

$$A \in \mathrm{D}(\mathbb{C}^2 \otimes \mathbb{C}^2), \qquad B \in \mathrm{D}(\mathbb{C}^2 \otimes \mathbb{C}^3), \qquad \text{and} \qquad C \in \mathrm{D}(\mathbb{C}^2 \otimes \mathbb{C}^4).$$

For each of the states A, B and C, compute its partial transpose (where the transpose is applied to the second register) and output the smallest eigenvalue of the resulting matrix. For each state, output whether the state is entangled or separable, or whether the partial transpose test was inconclusive.
Hint: Recall that the partial transpose test is always conclusive when the total dimension is at most 6.
(b) In the file `D.txt`, you will find the *pure* state $D \in \mathrm{D}(\mathbb{C}^2 \otimes \mathbb{C}^2)$. Calculate its entanglement entropy.