Quantum Information Theory, Spring 2020

Practice problem set #1

You do not have to hand in these exercises, they are for your practice only.

1. **Dirac notation quiz:** In the Dirac notation, every vector is written as a ‘ket’ $|\psi\rangle$ and every linear functional is written as a ‘bra’ $\langle \psi | = | \psi \rangle ^\dagger$, where $^\dagger$ denotes the adjoint. One can think of kets as column vectors and bras as row vectors. Hence, if $|\psi\rangle$ is a column vector, then $\langle \psi |$ denotes the row vector obtained by taking the conjugate transpose of the column vector.

   (a) Let $|\psi\rangle$ and $|\phi\rangle$ be vectors in $\mathbb{C}^n$ and $A$ an $n \times n$ matrix. Which of the following expressions are syntactically correct? For those that do, what kind of object do they represent (e.g., numbers, vectors, ...)? Can you write them using ‘ordinary’ notation?

   - i. $|\psi\rangle + \langle \phi |$
   - ii. $|\psi\rangle \langle \phi |$
   - iii. $A \langle \psi |$
   - iv. $\langle \psi | A$
   - v. $\langle \psi | A + \langle \phi |$
   - vi. $\langle \psi | \langle \phi | + A$
   - vii. $|\psi\rangle \langle \phi | A$
   - viii. $|\psi\rangle A \langle \phi |$
   - ix. $\langle \psi | A |\phi\rangle$
   - x. $\langle \psi | A |\phi\rangle + \langle \psi |\phi \rangle$
   - xi. $\langle \psi |\phi \rangle$
   - xii. $\langle \psi |\phi \rangle A$

   (b) Let $\rho = |\psi\rangle \langle \psi |$ and $\sigma = |\phi\rangle \langle \phi |$ be two pure states on the same system. Verify that

   \[ \text{Tr}[\rho \sigma] = |\langle \psi |\phi \rangle|^2. \]

   *Hint: You may use that the trace is cyclic, i.e. \( \text{Tr}[ABC] = \text{Tr}[CAB] = \text{Tr}[BCA] \).*

2. **Positive semidefinite operators:** Recall from class that an operator $A \in \text{L}(\mathcal{H})$ is called *positive semidefinite* if it is Hermitian and all its eigenvalues are nonnegative. We denote by $\text{PSD}(\mathcal{H})$ the set of positive semidefinite operators on a Hilbert space $\mathcal{H}$. Argue that the following conditions are equivalent:

   (a) $A$ is positive semidefinite.
   (b) $A = B^\dagger B$ for an operator $B \in \text{L}(\mathcal{H})$.
   (c) $A = B^\dagger B$ for an operator $B \in \text{L}(\mathcal{H}, \mathcal{K})$ and some Hilbert space $\mathcal{K}$.
   (d) $\langle \psi | A |\psi \rangle \geq 0$ for every $\psi \in \mathcal{H}$.
   (e) $\text{Tr}[AC] \geq 0$ for every $C \in \text{PSD}(\mathcal{H})$.

3. **Convexity:** Recall that a set $S$ is *convex* if $px + (1 - p)y \in S$ for every $x, y \in S$ and $p \in [0, 1]$.

   (a) Show that $\text{PSD}(\mathcal{H})$ is convex and closed under multiplication by $\mathbb{R}_{\geq 0}$ (i.e., a convex cone).
   (b) Show that $\text{D}(\mathcal{H})$ is convex.

4. **Positive semidefinite order:** Given two operators $A$ and $B$, we write $A \preceq B$ if the operator $B - A$ is positive semidefinite. Show that the following three conditions are equivalent:

   (a) $0 \preceq A \preceq I$.
   (b) $A$ is Hermitian and has eigenvalues in $[0, 1]$.
   (c) $\langle \psi | A |\psi \rangle \in [0, 1]$ for every unit vector $|\psi\rangle \in \mathcal{H}$. 

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5. **Bloch sphere**: Recall from the lecture that the state $\rho$ of a single qubit can be parameterized by the Bloch vector $\vec{r} \in \mathbb{R}^3$, $\|\vec{r}\| \leq 1$. Namely:

$$\rho = \frac{1}{2} [I + r_x X + r_y Y + r_z Z].$$

(a) Show that $r_x = \text{Tr}[\rho X]$, $r_y = \text{Tr}[\rho Y]$, and $r_z = \text{Tr}[\rho Z]$.

(b) Let $\sigma$ be another qubit state, with Bloch vector $\vec{s}$. Verify that $\text{Tr}[\rho \sigma] = \frac{1}{2} (1 + \vec{r} \cdot \vec{s})$.

(c) Let $\{|\psi_i\rangle\}_{i=0,1}$ denote an orthonormal basis of $\mathbb{C}^2$, $\mu: \{0,1\} \rightarrow \text{PSD}(\mathbb{C}^2)$ the corresponding basis measurement (i.e., $\mu(i) = |\psi_i\rangle\langle\psi_i|$ for $i \in \{0,1\}$), and $\vec{r}_i$ the Bloch vector of $|\psi_i\rangle\langle\psi_i|$. Show that the probability of obtaining outcome $i \in \{0,1\}$ when measuring $\rho$ using $\mu$ is given by $\frac{1}{2} (1 + \vec{r} \cdot \vec{r}_i)$. Show that $\vec{r}_0 = -\vec{r}_1$. How can you visualize these two facts on the Bloch sphere?

(d) Now imagine that $\rho$ is an unknown qubit state $\rho$ whose Bloch vector $\vec{r}$ you would like to characterize completely. Consider the following measurement with six outcomes:

$$\mu: \{(x,y,z) \times \{0,1\}\} \rightarrow \text{PSD}(\mathbb{C}^2), \quad \mu(a,b) = \frac{1}{6} [I + (-1)^b \sigma_a],$$

where $\sigma_x = X$, $\sigma_y = Y$, and $\sigma_z = Z$ are the three Pauli matrices. Show that $\mu$ is a valid measurement and that the probabilities of measurement outcomes are given by

$$p(a,b) = \frac{1}{6} [1 + (-1)^b r_a].$$

How can you visualize this formula on the Bloch sphere? Describe how measuring many copies of $\rho$ by using $\mu$ allows for estimating the entries of $\vec{r}$ to arbitrary accuracy.