1. **Quantum mutual information**: From class, we know that $I(A : B) \leq 2 \log d$ for every state $\rho_{AB} \in D(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^d$. Show that $I(A : B) = 2 \log d$ if and only if $\rho_{AB}$ is a pure state with $\rho_A = \rho_B = I/d$ (such states are called maximally entangled). Write down the Schmidt decomposition of a general state of this form.

   *Hint: In the exercise class you gave simple proof of the above inequality.*

2. **Classical mutual information**: From class, we know that $I(X : Y) \leq \log d$ for every distribution $p_{XY} \in P(\Sigma_X \times \Sigma_Y)$ with $|\Sigma_X| = |\Sigma_Y| = d$. Show that $I(X : Y) = \log d$ if and only if $p_{XY}(x,y) = \frac{1}{d} \delta_{f(x),y}$ for a bijection $f : \Sigma_X \to \Sigma_Y$ (such $p_{XY}$ are called maximally correlated).

   *Hint: In the exercise class you characterized the probability distributions with $H_{XY} = H_X$.*

3. **Entropic uncertainty relation**: Here you can prove another uncertainty relation. Let $\rho \in D(\mathbb{C}^2)$ and denote by $p_{\text{Std}}$ and $p_{\text{Had}}$ the probability distributions of outcomes when measuring $\rho$ in the standard basis and Hadamard basis, respectively. You will show:

   $$H(p_{\text{Std}}) + H(p_{\text{Had}}) \geq H(\rho) + 1 \quad (1)$$

   (a) Why is it appropriate to call (1) an **uncertainty relation**?

   (b) Find a state $\rho$ for which the uncertainty relation is saturated (i.e., an equality).

To start, recall the Pauli matrices $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   (c) Verify that $\frac{1}{2} (\rho + Z \rho Z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and deduce that $H(p_{\text{Std}}) = H(\frac{1}{2} (\rho + Z \rho Z))$.

   (d) Show that, similarly, $H(q_{\text{Had}}) = H(\frac{1}{2} (\rho + X \rho X))$. *Hint: $|\pm\rangle$ is the eigenbasis of $X$.*

Now consider the following three-qubit state,

$$\omega_{ABC} = \frac{1}{4} \sum_{a=0}^{1} \sum_{b=0}^{1} |a\rangle \langle a| \otimes |b\rangle \langle b| \otimes X^a Z^b \rho Z^b X^a,$$

where we denote $X^0 = 1$, $X^1 = X$, $Z^0 = 1$, $Z^1 = Z$. Note that subsystems A & B are classical.

   (e) Show that $H(ABC) = 2 + H(\rho)$. Use parts (c) and (d) to verify that $H(AC) = 1 + H(p_{\text{Std}})$, $H(BC) = 1 + H(p_{\text{Had}})$, and $H(C) = 1$ in state $\omega_{ABC}$.

   *Hint: Use the formula for the entropy of classical-quantum states that you proved last week.*

   (f) Use part (e) and the strong subadditivity inequality to deduce (1).
4. **(2 bonus points) Practice:** In this problem, you can explore the properties of typical subspaces. Consider the qubit state \( \rho = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |+\rangle \langle +|, \) where \(|+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle).\)

(a) Compute the largest eigenvalue \( \lambda \) as well as the von Neumann entropy \( H(\rho) \) of \( \rho \).

(b) Plot the following functions of \( k \in \{0, 1, \ldots, n\} \) for \( n = 100 \) as well as for \( n = 1000 \):

\[
d(k) = \binom{n}{k}, \quad r(k) = \frac{1}{n} \log \binom{n}{k}, \quad q(k) = \binom{n}{k} \lambda^k (1 - \lambda)^{n-k}
\]

(c) Plot the following functions of \( n \in \{1, \ldots, 1000\} \) for \( \epsilon = 0.1 \) as well as for \( \epsilon = 0.01 \):

\[
r(n) = \frac{1}{n} \log \text{dim } S_{n,\epsilon}, \quad p(n) = \text{Tr}[\Pi_{n,\epsilon} \rho \otimes n],
\]

where \( \Pi_{n,\epsilon} \) denotes the orthogonal projection onto the typical subspace \( S_{n,\epsilon} \) of \( \rho \).