Quantum Information Theory, Spring 2020

Practice problem set #8

You do not have to hand in these exercises, they are for your practice only.

1. Warmup:
   1. Show that, if ρ and σ are both pure states, D(ρ||σ) ∈ [0, ∞).
   2. Find a state ρ and a channel Φ such that H(Φ[ρ]) < H(ρ).
   3. Compute the relative entropy D(ρ||σ) for ρ = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| and σ = \frac{1}{4}|+\rangle\langle +| + \frac{3}{4}|−\rangle\langle −|.

2. Matrix logarithm: Recall that the logarithm of a positive definite operator with eigendecomposition Q = \sum_{i} \lambda_{i} |e_{i}\rangle\langle e_{i}| is defined as log(Q) = \sum_{i} log(\lambda_{i}) |e_{i}\rangle\langle e_{i}| (as always, our logarithms are to base 2). Verify the following properties:
   1. log(cI) = log(c)I for every c ≥ 0.
   2. log(Q \otimes R) = log(Q) \otimes I_{B} + I_{A} \otimes log(R) for all positive definite Q ∈ L(\mathcal{H}_{A}), R ∈ L(\mathcal{H}_{B}).
   3. log(\sum_{x} p_{x} |x\rangle\langle x| \otimes \rho_{X}) = \sum_{x} log(p_{x}) |x\rangle\langle x| \otimes I_{B} + \sum_{x} log(\rho_{X}) for every ensemble \{p_{x}, \rho_{X}\}_{x} of positive definite operators \rho_{X} ∈ D(\mathcal{H}_{B}).

Warning: It is in general not true that log(QR) = log(Q) + log(R)!

3. From relative entropy to entropy and mutual information: Use Problem 2 to verify the following claims from class:
   1. D(ρ||\frac{1}{d}I) = log d − H(ρ) for every ρ ∈ D(\mathcal{H}), where d = dim(\mathcal{H}).
   2. D(ρ_{AB}||ρ_{A} \otimes ρ_{B}) = I(\mathcal{A} : \mathcal{B})_{ρ_{AB}} for every ρ_{AB} ∈ D(\mathcal{H}_{A} \otimes \mathcal{H}_{B}), where ρ_{A} = Tr_{B}[ρ_{AB}] and ρ_{B} = Tr_{A}[ρ_{AB}]. You may assume that all three operators are positive definite.

4. Entropy and ensembles: In this problem, you will prove the upper bound on the Holevo information that we discussed in class: For every ensemble \{p_{x}, \rho_{X}\},

   \chi(\{p_{x}, \rho_{X}\}) ≤ H(p) \quad or, equivalently, \quad H(\sum_{x} p_{x} \rho_{X}) ≤ H(p) + \sum_{x} p_{x} H(\rho_{X}).

Moreover, show that equality holds if and only if the \rho_{X} with p_{x} > 0 have pairwise orthogonal image.

In terms of the cq-state corresponding to the ensemble, the above inequality can also be written as H(XB) ≥ H(B). This confirms our claim in Lecture 7. In Homework Problem 6.4 you showed that H(XB) ≥ H(X), but since the situation is not symmetric (X is classical but B is not) we now need a different argument.

1. First prove these claims assuming that each \rho_{X} is a pure state, i.e., \rho_{X} = |\psi_{X}\rangle\langle \psi_{X}|.
   
   Hint: Consider the pure state |\Phi\rangle = \sum_{x} \sqrt{p_{x}} |x\rangle \otimes |\psi_{X}\rangle and compare the entropy of the first system before and after measuring in the standard basis.

2. Now prove the claims for general \rho_{X}.
   
   Hint: Apply part (a) to a suitable ensemble obtained from the eigendecompositions of the \rho_{X}.