State conversion by LOCC (specifically, $|WAB\rangle \rightarrow |WAB\rangle$).

**Marginalization and wealth inequality**

$\sum_{i} p(i) = 0$ - wealth of person $i$

$\frac{\sum p(i)}{\sum_{i} p(i)}$ - how unequal is the distribution $p$?

Clearly, $p = (1, 0, \ldots, 0)$ is the most unequal.

Also, $q = (\frac{1}{n}, \ldots, \frac{1}{n})$ is the most equal.

Robin Hood move:

$M(c) = c \sum X + (1-c) X$

$c \in [0,1]$

$\begin{pmatrix} c & 1-c \\ 1-c & c \end{pmatrix}$

$p \rightarrow q = M(c) . p$

If $c \in (0,1)$, this always moves the distribution more equal.
Def. \( A \in L(C^\Sigma) \)

- **A** is **stochastic** if
  1. \( A_{ij} \geq 0 \quad \forall i,j \in \Sigma \)
  2. \( \sum_{i} A_{ij} = 1 \quad \forall i \in \Sigma \)

- **A** is **doubly stochastic** if
  1. **A** is stochastic and
  2. \( \sum_{j} A_{ij} = 1 \quad \forall i \in \Sigma \)

- **A** is a **permutation** if **A** is doubly stochastic and
  4. \( A_{ij} \in \{0,1\} \quad \forall i,j \in \Sigma \)

Stochastic \( \Leftrightarrow \) map prob. distributions (prob. channel) to prob. dists.

Permutations \( \Leftrightarrow \) stochastic and inverse also (prob. unitaries) \( \exp(\Sigma) \) stochastic.

Any convex combination of permutation matrices is doubly stochastic.
Converse:

Theorem (Birkhoff - von Neumann)

A \in L(\mathbb{R}^\mathbb{S}) is doubly stochastic iff there exists a prob. distri. \mu on \text{Sym}(\mathbb{S}) s.t.

\[ A = \sum_{\pi} \mu(\pi) V_\pi \]

where \( V_\pi \in L(\mathbb{R}^\mathbb{S}) \) is the permutation matrix

for \( \pi : (V_\pi)_{ij} = \delta_{i \pi(j)} \).

Fact. Any doubly stochastic matrix can be obtained by a sequence of Robin Hood moves.

Def. Let \( u, v \in \mathbb{R}^\mathbb{S} \). Then \( u \) majorizes \( v \) if \( v = Au \), for some doubly stochastic \( A \).

Notation: \( u \succ v \) or \( v \preceq u \).

Let's sort \( u \) (reverse order) by defining

\[ v_1(u) \succ v_2(u) \succ \cdots \succ v_m(u) \]

and \( \sum_{i=1}^m v_i(u) = 1 \).

Theorem. \( u, v \in \mathbb{R}^\mathbb{S} \). Then \( v \preceq u \) iff

\[ \sum_{i=1}^m v_i(u) \leq \sum_{i=1}^m v_i(u) \quad \text{and} \quad \sum_{i=1}^m v_i(u) = \sum_{i=1}^m v_i(u) \quad \forall m \in \mathbb{E}, n \in \mathbb{N} \].
Majorization for Hermitian operators

The quantum version of permutation is a unitary matrix. And doubly stochastic matrices become

Def: $\Phi \in C(H)$ is mixed-unitary if

$$\Phi(M) = \sum_{i \in \mathcal{E}} p(i) U_i M U_i^\dagger$$

for some $p \in P(\Sigma)$ and $U_i \in U(H)$.

Def: Let $A$ and $B$ be Hermitian ops on $H$. Then $A$ majorizes $B$ iff $B = \Phi(A)$ for some mixed-unitary channel $\Phi \in C(H)$.

Notation: $A \succ B$ or $B \prec A$.

Theorem (Wielandt) $B \prec A$ iff $\lambda(B) \prec \lambda(A)$, where $\lambda(A) \in \mathbb{R}$ is the spectrum of $A$.

---

Alice  Bob

1. $U_{AB} \triangleright$

    \[ \xymatrix{ 1 & 1 \ar@{<->}[r] & \text{LOCC} } \]

    Can they solve this by LOCC?
Theorem (Nielsen)

Let \( |U_{AB} \rangle \otimes |U_{AB} \rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \). The following are equivalent:

1. \( T^1 \mathbb{B}[11] \mathbb{X} \mathbb{X}^1 \mathbb{B}[11] \mathbb{X} \mathbb{X}^1 \)

2. \( \exists \mathbb{E} \mathbb{E}[11] \mathbb{X} \mathbb{X}^1 \mathbb{E} \mathbb{E} \) for some one-way LOCC protocol \( \mathbb{E} \in \text{LOCC}(A:B) \) from Alice to Bob.

3. Same, but \( \mathbb{E} \) is one-way from Bob to Alice.

4. Same, but \( \mathbb{E} \in \text{SepC}(A:B) \).

Proof (1 \( \Rightarrow \) 2) Exercise: \( L, R \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B) \)

\[ T^1 \mathbb{A}[11] \mathbb{X} \mathbb{X}^1 = LR^+ \in \mathcal{L}(\mathcal{H}_B) \]

If \( L = 11 \mathbb{x} \mathbb{x} \) then \( |L \rangle \rangle = 1o \rangle \rangle \otimes 1b \rangle \rangle \).

Let \( X, Y \in \mathcal{L}(\mathcal{H}_A, \mathcal{H}_B) \) s.t. \( |X \rangle = |u \rangle \)

Then 1. is equivalent to

\[ XX^+ \succ YY^+ \]

By 2., there exists a mixed-unitary channel s.t.

\[ XX^+ = \sum \rho (\xi) W \odot YY^+ W^+ \]

\[ \xi \in \Sigma \]

\[ W \in \text{SepC}(X_{AB}) \]

\[ \therefore \sum \rho (\xi) (W \odot YY^+ W^+) = ZZ^+ \]
Given \(XX^+ = \Xi_\Xi\Xi^+\), how one \(X\) and \(\Xi\) related?

The singular value decomposition:

\[
X = \sum_{k=1}^{r} s_k \langle x_k, y_k \rangle \chi_{k} \quad r = \text{rank}(X)
\]

\[
= \sum_{k=1}^{r} s_k \langle x_k, y_k \rangle \chi_{k} \quad \text{(orthonormal)}
\]

\[
XX^+ = \sum_{j,k=1}^{r} s_j s_k \langle x_j, x_k \rangle \langle y_j, y_k \rangle \chi_{j} \chi_{k} = \Xi_\Xi \Xi^+
\]

Conclude that the singular value dec. of \(X\)

\[
\Xi = \sum_{k=1}^{r} s_k \langle x_k, y_k \rangle \chi_{k} \quad \text{(orthonormal)}
\]

For some orthonormal set \(\{w_k\}_k : k = 1, \ldots, r\).

Define \(V | y_k \rangle = |w_k \rangle\), so that

\[
XV^+ = \Xi = \sum_{i \in S} \delta_{ij} (w_i, y) \otimes w_i
\]
Let \( |w\rangle = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} i |i\rangle \).

\[ \text{Tr}_B \left( |w\rangle \langle w| \otimes 1 \right) = \frac{1}{n} \sum_{i=1}^{n} i |i\rangle \langle i|, \]

spectrum of this is \( q = (\frac{1}{n}, \ldots, \frac{1}{n}) \).

\( q \) is a spectrum of \( \text{Tr}_B \left( |w\rangle \langle w| \right) \).

Up always true!

\( \Rightarrow \) The maximally entangled state can be converted to any other pure state by one-way LOCC.
If \( p = (1, 0, \ldots, 0) \)
\[ q = \left( \frac{1}{n}, \ldots, \frac{1}{n} \right) \]
then for any \( s \), a prob. dist.

\[ q \ll s \]

In particular, \( q \ll p \).