1. (4 points) **Rényi-2 entropy**: In this problem you will study a new entropy measure called the Rényi-2 entropy. It is defined by $H_2(\rho) := -\log \text{Tr}[\rho^2]$ for any quantum state $\rho \in D(\mathbb{C}^d)$.

(a) Find a formula for $H_2(\rho)$ in terms of the eigenvalues of $\rho$.
(b) Show that $H_2(\rho) \leq H(\rho)$ by using Jensen’s inequality.
(c) Show that $\text{Tr}[\rho^2] = \text{Tr}[F \rho \otimes |i\rangle \langle j| \otimes |i\rangle \langle j|]$ for all $i, j \in \{1, \ldots, d\}$, is the swap operator.

2. (4 points) **Average entanglement**: In this exercise you will study the average entanglement of a random pure state in $\mathcal{H}_A \otimes \mathcal{H}_B$ drawn from the uniform distribution $d\psi_{AB}$ discussed in class.

Recall that the entanglement entropy of a pure state $|\psi_{AB}\rangle$ is given by $H(\rho_A) = H(\rho_B)$, where $\rho_A$ and $\rho_B$ are the reduced states of $|\psi_{AB}\rangle$.

(a) Let $F_{AA}, F_{BB}$ denote the swap operators on $\mathcal{H}_A \otimes \mathcal{H}_A$ and let $d_A = \dim \mathcal{H}_A, d_B = \dim \mathcal{H}_B$. Use the integral formula for the symmetric subspace to deduce that

$$\int |\psi_{AB}\rangle \langle \psi_{AB}| \otimes 2^2 d\psi_{AB} = \frac{1}{d_A d_B (d_A d_B + 1)} (I_{AA} \otimes I_{BB} + F_{AA} \otimes F_{BB}).$$

(b) Verify that $\int \text{Tr}[\rho_A^2] d\psi_{AB} = \frac{d_A^2 + d_B^2}{d_A d_B + 1}$.
(c) Show that the average Rényi-2 entropy $H_2(\rho_A)$ for a random pure state $|\psi_{AB}\rangle$ is at least $\log(\min(d_A, d_B)) - 1$. Conclude that the same holds for the entanglement entropy.

*Hint: Use Problem 1 and Jensen’s inequality.*

3. (4 points) **Haar measure**: In the exercise class, we discussed the Haar measure on $U(\mathcal{H})$, which is the unique probability measure $dU$ with the following property: For every continuous function $f$ on $U(\mathcal{H})$ and for all unitaries $V, W \in U(\mathcal{H})$, it holds that $\int f(U) dU = \int f(VUW) dU$.

(a) Argue that, for any operator $A \in L(\mathcal{H} \otimes \mathcal{H})$, the so-called twist $\int U \otimes U A U^\dagger \otimes U dU$ can always be written as a linear combination of permutation operators $R_{\pi}, \pi \in S_n$.

(b) Deduce that $\int U \otimes U U^\dagger \otimes U dU = \alpha I + \beta F$ for every $A \in L(\mathcal{H} \otimes \mathcal{H})$, where $F$ is the swap operator on $\mathcal{H} \otimes \mathcal{H}$, $\alpha = \frac{d}{d^2 - d} \text{Tr}[A] - \frac{1}{d^2 - d} \text{Tr}[FA]$, and $\beta = \frac{d}{d^2 - d} \text{Tr}[FA] - \frac{1}{d^2 - d} \text{Tr}[A]$. 

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