Quantum Information

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Seek to leverage laws of QM for information processing...

Quantum Information

communication  cryptography

networks  algorithms

quantum bits  computation  complexity

entropy  entanglement  error correction

tensor networks  quantum simulation

...but also toolbox and language for studying q. many-body systems.
Physics vs Information: Thermodynamics

Irreversibility (2\textsuperscript{nd} law) vs coarse graining

Thermodynamics of computation: Cost of erasing a bit?

\[ W \geq kT \ln(2) \]

Most logic gates are irreversible. Is there a fundamental cost to computing? No!

Bennett (1973): Efficient reversible computing is possible!
Simulating quantum physics difficult for classical computers. Hilbert space is exponentially large

Why don’t we build a quantum computer? Feynman, Deutsch, ...

Shor’s algorithm (1984): quantum computers may offer vast speedups for classical problems

Google “quantum supremacy” experiment (2019)

Today, quantum simulation still one of most promising applications.
Physics vs Information: Language and Toolbox

Quantum information is different: No cloning, uncertainty principle, Bell violations, entanglement, decoherence, ...

QIT offers language and toolbox to study and exploit these phenomena. Examples:

- Uncertainty principle $\Rightarrow$ quantum cryptography
- Bell violations $\Rightarrow$ device-independent control
- Entanglement $\Rightarrow$ many-body physics

In recent years, exciting research at interface of quantum information with QFT and gravity.
Plan

Goal: Discuss language, toolbox, key concepts of quantum information. Survey applications to holography.

Today: States, Channels, Entropy, Entanglement
Tue: Entanglement in QFT and Holography
Wed: Toy Models of Holography, Decoupling, Black Holes
Thu: Tensor Network Models, Error Correction

“Homework” and open problems throughout.
“The relation between information theoretical concepts in CFT and geometric concepts in AdS has taught us many lessons.”
Interrupt me!

If too slow (or too fast), please let me know. 😊

If not detailed enough, please ask. 😊
1. States, Channels, Entropy

Literature: Lectures on “Symmetry and Quantum Information”
(https://staff.fnwi.uva.nl/m.walter/qit18/)
Quantum states

Density operators on Hilbert space:

\[ \rho = \sum_x p_x \left| \psi_x \right\rangle \left\langle \psi_x \right| \]

Pure states: \( \rho = \left| \psi \right\rangle \left\langle \psi \right| \)

Mixed states model ensembles \( \{p_i, \rho_i\} \):

\[ \rho = \sum_i p_i \rho_i \]

States of qubit: Bloch ball
Entropy

Von Neumann entropy:

\[ S(\rho) = -\text{tr} \rho \log \rho \]

only depends on nonzero eigenvalues: \( S(\rho) = S(U\rho U^+) \)

\[ 0 \leq S(\rho) \leq \log(d) \]

pure \( \rho = I/d \)

Modular Hamiltonian:

\[ K_\rho = -\log \rho \]

state-dependent, often nonlocal

“First law of entanglement”

\[ S(\rho + \delta\rho) = S(\rho) + \text{tr}[\delta\rho K_\rho] + \ldots \]

Proof? Homework!
Renyi entropies and replica trick

Von Neumann entropy often difficult to compute ⇒ Renyi entropies:

$$S_n(\rho) = \frac{1}{1-n} \log \text{tr}[\rho^n]$$

$$= (1-n)^{-1} \log \sum_x p_x^n$$

- $$S_0(\rho) = \log \text{#nonzero eigenvalues}$$
- $$S_1(\rho) = S(\rho)$$
- $$S_2(\rho) = -\log \text{tr}[\rho^2]$$

$$\log(d) \geq S_0(\rho) \geq S(\rho) \geq S_2(\rho) \geq \ldots \geq 0$$

equal if $\rho$ flat spectrum

Easy to calculate for integer $n>1$:

$$\text{tr}[\rho^2] = \text{tr}[\rho \otimes 2 F]$$

where

$$F |xy> = |yx>$$

swap trick

$$\text{tr}[\rho^n] = \text{tr}[\rho \otimes n C_n]$$

where

$$C_n |x_1x_2\ldots> = |x_2x_3\ldots x_1>$$

Proof? Just expand it.
Joint systems

Reduced states of global states $\rho_{AB}$ are given by partial trace:

$$\rho_A = \text{tr}_B(\rho_{AB})$$
$$\langle a|\rho_A|a'\rangle = \sum_b \langle a b|\rho_{AB}|a'b\rangle$$

Maximally entangled state (Bell/EPR pair):

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$S_{AB} = \frac{1}{2} \left( |00\rangle|00\rangle + |10\rangle|10\rangle + |01\rangle|01\rangle + |11\rangle|11\rangle \right)$$

$$S_A = \frac{1}{2} \left( |0\rangle|0\rangle + |1\rangle|1\rangle \right) = \frac{1}{2}$$

maximally mixed

Thus, pure states often have mixed reduced states. Conversely:

Any state $\rho_A$ has a purification $\rho_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$. 
Correlations

We say that a state is *correlated* if not a product:

\[ \rho_{AB} \neq \rho_A \otimes \rho_B \]

\[ \langle O_A O'_B \rangle \neq \langle O_A \rangle \langle O'_B \rangle \]

for some pair of observables

Correlations can have *quantum* or *classical* origin:

Maximally entangled state:

\[ |\Phi^+_AB\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]

Max. classically correlated:

\[ \gamma_{AB} = \frac{1}{2} (|00\rangle\langle 00| + |11\rangle\langle 11|) \]

In both cases, \( \rho_A = \rho_B = I/2 \), but \( \rho_{AB} \neq I/4 \).

How to quantify correlations?
Mutual information

**Mutual information:**

\[
I(A:B) = S(A) + S(B) - S(AB) \geq 0
\]

\[= 0 \text{ iff product}\]

\[I(A:B) = 2 \log(d) \text{ iff maximally entangled} \]

\[I(A:B) = \log(d) \text{ if max. classical correlated}\]

**Pinsker's inequality** bounds correlation functions:

\[
\left| \langle O_A O'_B \rangle - \langle O_A \rangle \langle O'_B \rangle \right| \leq ||O_A|| ||O'_B|| \sqrt{2 \ln(2)} I(A:B)
\]

**Strong subadditivity (SSA):**

\[I(A:BC) \leq I(A:B)\]

never more correlated with subsystem

Fundamental, intuitive, difficult to prove.
Quantum channels

What are the most general transformation of quantum states?

\[ \rho \rightarrow \rho' \]

**Quantum channel**: Any combination of unitary evolution, partial traces, adding auxiliary systems.

Mathematically: Completely positive trace-preserving maps.

**Data processing inequality:**

\[ I(A:B) \geq I(A':B') \]

...if \( \rho_{A'B'} \) obtained from \( \rho_{AB} \) by quantum channels \( A\rightarrow A', B\rightarrow B' \).

Homework: Prove this using SSA.
Application: Holevo bound

How many bits can we communicate by sending 1 qubit?

Sender

$\{0,1\}^n \ni x \xrightarrow{\text{encoder}} \rho(x) \xrightarrow{\text{qubit state}} y \xrightarrow{\text{decoder}}$ Receiver

Challenge: Do not know optimal states nor optimal decoder!

$$\rho_{XB} = 2^{-n} \sum_x |x\rangle\langle x| \otimes \rho(x) \quad \Rightarrow \quad \rho_{XY} = 2^{-n} \sum_x |xx\rangle\langle xx|$$

...if can decode perfectly. Using the data processing inequality:

$$n = I(X:Y) \leq I(X:B) = H(B) - \sum_x p_x H(\rho(x)) \leq \log 2 = 1$$

Homework: Verify this.

1 bit/qubit $\Rightarrow$ no quantum advantage!
2. Entanglement

Literature: Lectures on “Symmetry and Quantum Information” (https://staff.fnwi.uva.nl/m.walter/qit18/)
We say that a state is **separable** if mixture of product states:

\[ \rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \]

Otherwise, the state is called **entangled**.

**Separable** states are precisely those that can be created by **Local Operations and Classical Communication (LOCC)**.

Motivation: classical correlations ≠ entanglement.

That is, to create **entanglement** need to send **quantum systems**.

**Homework:** Show this.
Entanglement in pure states

For pure states, the situation simplifies.

\(|\Psi_{AB}\rangle\) is \textit{entangled} if not a product:

\[ |\Psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\phi_B\rangle \]

That is, all correlations in pure states boil down to entanglement.

[Headrick]
Schmidt decomposition

\[ |\Psi_{AB}\rangle = \sum_{i=1}^{r} s_i \left| e_i \right\rangle \otimes \left| f_i \right\rangle \]

\[ \rho_A = \sum_{i=1}^{r} s_i^2 \left| e_i \right\rangle \langle e_i \right| \]

\[ \rho_B = \sum_{i=1}^{r} s_i^2 \left| f_i \right\rangle \langle f_i \right| \]

- Reduced states have same eigenvalues, entropies, ... and characterize entanglement:

\[ |\Psi_{AB}\rangle \text{ product } \iff r = 1 \iff \rho_A \text{ pure } \iff \rho_B \text{ pure} \]

- Any two purifications of \( \rho_A \) are related by isometry on B
Extensions and Monogamy

Even if $\rho_{AB}$ mixed: $\rho_A$ pure $\Rightarrow$ $\rho_{AB} = \rho_A \otimes \rho_B$

Take purification $|\Psi_{ABC}\rangle$ of $\rho_{AB}$. Since $\rho_A$ pure, $|\Psi_{ABC}\rangle = |\Psi_A\rangle \otimes |\Psi_{BC}\rangle$.

This implies that pure state entanglement is monogamous:

$AB$ pure $\Rightarrow AB$ uncorrelated with $C$

$\textbf{Monogamy: } AB$ and $AC$ cannot both be pure entangled.

In contrast, classical correlations can be arbitrarily shared.
Entanglement entropy

Schmidt decomposition suggests to quantify entanglement by the entropy of reduced states → **Entanglement entropy**:

\[ 0 \leq S_E = S(A) = S(B) \leq \log d_A \leq \log d_B \]

- **product state**: maximally entangled
- **maximally entangled**: product state

**Interpretation**: Optimal conversion rate with Bell pairs:

\[ |\Psi_{AB}\rangle \otimes^n \xrightarrow{\text{LOCC}} (|00\rangle + |11\rangle) \otimes S_{E n} \]

- Entanglement transformations “reversible”
- Bell pairs = unit of entanglement

\[ \text{n} \rightarrow \infty \text{ copies, error} \rightarrow 0 \]

\[ \otimes S_{E n} \]

\[ \text{for pure states} \]
Suppose a black hole is created from infalling matter and we watch it evaporate.

\[ R = \text{Hawking radiation emitted up to some time} \]
\[ B = \text{black hole = later Hawking radiation} \]

A semiclassical calculation suggests entropy of radiation increases until the end. But in a unitary theory, radiation will be pure once BH has evaporated...

Intuitively, early radiation is entangled with black hole, while late radiation is entangled with early radiation.
Application: Page curve

Simplest toy model: Assume that evaporation described by random unitary evolution.

\[ |\Psi_{BR}\rangle = \text{random pure state} \]

\[ b = \log d_B \]

\[ r = \log d_R \]

Page's theorem: For typical states,

\[ S_E = \min(b,r) - O(1) \]

almost maximal!

It would be more physical to consider a random state in a fixed total energy subspace or a random Hamiltonian evolution.
Derivation of the Page formula

Idea: Lower-bound average Renyi-2 entropy $S_2(R)$ using swap trick.

Key formula:

$$\Psi^\otimes_2 = \frac{I + F}{d(d + 1)}$$
for random $\Psi = |\Psi\rangle\langle\Psi|$

Apply this to $|\Psi\rangle = |\Psi_{BR}\rangle$:

$$\Psi_{BR}^\otimes_2 = \frac{I_{BB} \otimes I_{RR} + F_{BB} \otimes F_{RR}}{d_B d_R (d_B d_R + 1)}$$

$$\Rightarrow \quad \text{tr} \Psi_R^2 = \text{tr} \Psi_{BR}^\otimes_2 F_{RR} \leq \frac{\text{tr} (I_{BB} \otimes F_{RR} + F_{BB} \otimes I_{RR})}{d_B^2 d_R^2} = \frac{1}{d_R} + \frac{1}{d_B}$$

swap trick

$$\Rightarrow \quad S_2(R) \geq -\log \text{tr} \Psi_R^2 \geq -\log \left(\frac{1}{d_R} + \frac{1}{d_B}\right) \geq \min(b, r) - 1$$

Jensen’s inequality

Homework: Verify this.
Entanglement as a resource

What is entanglement good for? **Four examples** where entanglement enables otherwise impossible capabilities:

1) **Superdense coding**: communicate 2 bits by sending 1 qubit

   Holevo bound shows that impossible w/o entanglement

2) **Teleportation**: communicate 1 qubit by sending 2 bits

3) **Violating Bell inequalities**: produce non-classical correlations

4) **Quantum cryptography**: distill a shared secret key

It is also **necessary** for any quantum computational speedup.
Superdense coding

If Alice and Bob share EPR pair, they can use it to communicate 2 bits by sending 1 qubit!

\[
\begin{align*}
|\Phi_{AB}^{(00)}\rangle &= (|00\rangle + |11\rangle) / \sqrt{2} = (I \otimes I)|\Phi_{AB}^{+}\rangle \\
|\Phi_{AB}^{(01)}\rangle &= (|00\rangle - |11\rangle) / \sqrt{2} = (Z \otimes I)|\Phi_{AB}^{+}\rangle \\
|\Phi_{AB}^{(10)}\rangle &= (|10\rangle + |01\rangle) / \sqrt{2} = (X \otimes I)|\Phi_{AB}^{+}\rangle \\
|\Phi_{AB}^{(11)}\rangle &= (|10\rangle - |01\rangle) / \sqrt{2} = (XZ \otimes I)|\Phi_{AB}^{+}\rangle
\end{align*}
\]

4 orthogonal states created by local operation

"Bell basis"

"beating" the Holevo bound!
Teleportation

If Alice and Bob share EPR pair, they can use it to communicate 1 qubit by sending 2 bits!

Why does it work? If outcome $x=z=0$, post-measurement state:

$$\psi = \left(\langle \Phi^+_M | \otimes I_B \right)(\psi_M \otimes \Phi^+_A)$$

$$= \frac{1}{2} \sum_{j,k} (\langle j |_M \otimes \langle j |_A \otimes I_B \rangle)(\psi_M \otimes |k\rangle_A \otimes |k\rangle_B)$$

$$= \frac{1}{2} I_{M\rightarrow B} |\psi_M\rangle = \frac{1}{2} |\psi_B\rangle$$
Nonlocal correlations

Alice and Bob play CHSH game:

- Winning condition:
  - $x$ and $y$ are inputs to Alice and Bob, respectively.
  - The referee determines the output based on $a(x)$ and $b(y)$.
  - The output is $a \oplus b$.

Local classical strategy: $a = a(x)$, $b = b(y)$

$$a(0) \oplus b(0) \oplus a(0) \oplus b(1) \oplus a(1) \oplus b(0) \oplus a(1) \oplus b(1) \equiv 0$$

$\Rightarrow$ will get at least one answer wrong: $P_{\text{win}} \leq \frac{3}{4}$

This is a Bell inequality – a bound on classical correlations!
Nonlocality and quantum cryptography

If Alice and Bob share EPR pair, they can do better and achieve

\[
p_{\text{win}, q} = 0.85..\]

Tsirelson: optimal

strategy “unique” (rigidity)

\(\Rightarrow\) can certify entanglement from correlations alone!

Application: In **quantum key distribution**, Alice and Bob want to create a key **secret** from everyone else.

1) Play nonlocal game to ensure that state \(|\Phi_{AB}^+\rangle\) by **rigidity**
2) Then \(|\Psi_{ABE}\rangle = |\Phi_{AB}^+\rangle \otimes |\psi_E\rangle\) by **monogamy**
3) Now EPR pair measure to get random secret bit.
3. Entanglement in Mixed States

Literature: Lecture notes “Symmetry and Quantum Information”,
https://staff.fnwi.uva.nl/m.walter/qit18/
Entanglement in mixed states

Recall that a state is **separable** if mixture of product states:

\[
\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)}
\]

not canonical, typically non-orthogonal

**Bad news:** **NP-hard** to check if $\rho_{AB}$ separable

$\Rightarrow$ no entanglement measure is faithful and easy to compute

A practical problem – meaningful calculations are difficult.

Similarly, multipartite entanglement. $\rho_{AB}$ vs purification $|\Psi_{ABC}\rangle$
Bound entanglement

Can create any entangled state by LOCC given enough Bell pairs.

Bad news: Transformation usually irreversible.

There even exist “bound entangled” states such that no Bell pairs can be obtained from any number of copies!

Zoo of entanglement measures: entanglement cost $E_C$, distillable entanglement $E_D$, ...

Yet there are some practically useful criteria...
PPT criterion

Idea: Necessary for separability $\Leftrightarrow$ sufficient for entanglement

**Partial transpose (PT):**

\[
\langle ab | \rho_{AB}^\Gamma | a'b' \rangle = \langle ab' | \rho_{AB} | a'b \rangle
\]

“partial time reverse”

If $\rho_{AB}$ separable then $\rho_{AB}^\Gamma$ is again a density operator.

\[
\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \Rightarrow \rho_{AB}^\Gamma = \sum_i p_i \rho_A^{(i)} \otimes (\rho_B^{(i)})^T
\]

**PPT criterion:**

$\rho_{AB}^\Gamma$ negative eigenvalues $\Rightarrow$ $\rho_{AB}$ entangled

E.g. $\begin{pmatrix} \Psi & 0 \end{pmatrix} \otimes 1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix}$ $\Rightarrow$ \[
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \end{pmatrix}
\]
Negativity

Partial transpose has $tr=1$. Thus, has negative eigenvalues $\Leftrightarrow$ sum of absolute eigenvalues is $> 1$.

**Negativity:**

$$N(\rho) = \left(\sum_i |\lambda_i| - 1\right)/2$$

**Logarithmic negativity:**

$$E_N(\rho) = \log \sum_i |\lambda_i|$$

**How to calculate?**

1) Compute “Renyi negativities” $tr \ (\rho_{AB})^{2n}$ and let $n \to \frac{1}{2}$

2) Use replica trick: $tr \ (\rho_{AB})^{2n} = tr \ (\rho_{AB})^\otimes 2n \ (C^{2n} \otimes C^{-2n})$

⇒ **Feasible** in field theory and holography!

**Extendibility criterion**

Say $\rho_{AB}$ has *k-extension* if there is state $\sigma$ on $AB_1 \ldots B_k$ with

$$
\rho_{AB} = \sigma_{AB_1} = \cdots = \sigma_{AB_k}
$$

If $\rho_{AB}$ separable then has k-extension for all k.

$$
\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \Rightarrow \sigma_{AB_1 \ldots B_k} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \ldots \otimes \rho_B^{(i)}
$$

Conversely, if k-extension then $O(1/k)$ to separable.

**Criterion:** $\rho_{AB}$ separable $\iff$ has k-extension for all k

⇒ Entanglement is *monogamous* also for mixed state!
Bonus: De Finetti theorem

Suppose that $A_1...A_n$ is permutation-symmetric. Then reduced states are close to mixtures of product states:

De Finetti Theorem: $\rho_{A_1...A_k} \approx \int d\sigma \ p(\sigma) \ \sigma^{\otimes k}$ if $k \ll n$

e.g. $|00...0\rangle + |11...1\rangle$: any reduction is separable

- another version of monogamy
- justifies for why in mean field theory it suffices to consider product states
Bonus: Squashed entanglement

While mutual information is not a good entanglement measure, can construct one using the conditional mutual information:

\[ I(A:B|C) = I(A:BC) - I(A:C) \]

\[ = S(AC) + S(BC) - S(ABC) - S(C) \geq 0 \]

Squashed entanglement:

\[ E_{sq}(A:B) = \frac{1}{2} \min_{\rho_{ABC}} I(A:B|C) \]

Intuition: entanglement = correlations that cannot be shared

Properties:

1) \( 0 \leq E_{sq} \leq \frac{1}{2} I(A:B) \leq \log \min(d_A, d_B) \)
2) For pure states: \( E_{sq} = \frac{1}{2} I(A:B) = S_E \)
3) Separable \( \iff E_{sq} = 0 \)
4) Monogamy: \( E_{sq}(A:B) + E_{sq}(A:C) \leq E_{sq}(A:BC) \)

Homework: Show all but \( \in 3. \)
4. Entanglement in Field Theory

Quantum information & field theory

Do quantum information tools apply to quantum field theory?

Challenge: Basic notions such as subsystems, entanglement, entropy, ... more subtle!

Theoretical insights: c-theorem from strong subadditivity, Bekenstein bound from relative entropy, renormalization vs QEC...

Another motivation: Quantum computers can simulate quantum mechanics. Can we simulate QFTs or even quantum gravity...?
Subsystems in relativistic QFT

Causal domain of $A$:

\[ D(A) = \{ p : \text{every maximal causal curve through } p \text{ intersects } A \} \]

$\Sigma$ is Cauchy slice if acausal and $D(\Sigma) = \text{everything}$.

Time slice axiom:

\[ \Sigma \leftrightarrow \text{global state} \leftrightarrow \text{Hilbert space } H \]

\[ A \subseteq \Sigma \leftrightarrow \text{reduced state in } D(A) \leftrightarrow "H = H_A \otimes H_B" \]

$D(A) = D(A') \Rightarrow \rho_A \text{ and } \rho_{A'} \text{ should be unitarily related}$
Correlations in QFT

Consider e.g. free scalar field with mass $m$ in Minkowski space:

$$H = \int d^3x \ (\pi(x))^2 + (\nabla \phi(x))^2 + m^2 \phi(x)^2$$

$[\pi(x), \phi(y)] = i\delta^3(x-y)$

Correlation functions:

$$\langle \phi(x) \rangle = 0$$

$$\langle \phi(x) \phi(y) \rangle \propto \begin{cases} 
|x-y|^{-2} & \text{if } |x-y| \ll \xi \\
\exp(-|x-y|/\xi) & \text{if } |x-y| \gg \xi
\end{cases}$$

$\xi \sim 1/m$ correlation length

General form (short-distance power law, long-distance decay) believed to hold in any relativistic QFT. If $m=0$, decay can be power law.

Amusing to compare with Bell pair:

$$\langle X \rangle = \ldots = \langle Z \rangle = 0$$

$$\langle XX \rangle = \ldots = \langle ZZ \rangle = 1$$
Entanglement in QFT

Correlation functions:

\[ \langle \phi(x)\phi(y) \rangle \propto \begin{cases} 
|x-y|^{-2} & \text{if } |x-y| \ll \xi \\
\approx 0 & \text{if } |x-y| \gg \xi 
\end{cases} \]

Thus, might expect that entanglement entropy satisfies an area law:

\[ S(A) \propto |\partial A| / \varepsilon^{d-2} \]

More generally, might expect that all divergences arise from local integrals over entangling surface \( \partial A \).

That is, assuming \( \xi < \infty \). E.g. for CFTs in \( d=1+1 \), power law decay leads to \( \log(|A|/\varepsilon) \) divergence, as we will discuss momentarily.
**Entanglement in QFT**

Observables in A, B commute, but Hilbert space does **not** factorize.

\[ H \neq H_A \otimes H_B \]

(cf. divergence across entangling surface)

- Reduced states **not** described by density operators
- Entanglement entropies **not** obviously well-defined

What can be said rigorously? → algebraic QFT literature, Witten’s review

**Reeh-Schlieder:**

\[ \{ O_A |\Omega_{AB}\rangle \} \text{ dense} \]

Confusing? No, \( O_A \) will **not** be unitary!

Homework: Show that in finite dim any \( |\Psi_{AB}\rangle \) can be written as \( O_A |\Phi_{AB}^+\rangle \).

Relative entropies & various entanglement measures can be rigorously defined and computed/bounded → e.g., still makes sense to distill EPR pairs!

**Bisognano-Wichmann:** "modular Hamiltonian" of Rindler wedge
**Entanglement Entropy in QFT**

We will proceed cavalierly since we must anyways regulate entanglement entropy to obtain finite answer.

General strategy: UV regulate and compute *universal quantities*

- **Coefficient of** $\log(|A|/\varepsilon)$
  
- **Relative entropy**
  \[
  D(\rho||\sigma) = \text{tr} \rho \left( \log \rho - \log \sigma \right)
  \]

- **Mutual information** $I(A:B)$
  
  If $A$, $B$ don’t touch: “$H_{AB} = H_A \otimes H_B$”
  
  $\Rightarrow$ rigorously defined in QFT!

  Intuition: divergences cancel
Euclidean path integrals

Let us consider states that are prepared by Euclidean path integrals. E.g., unnormalized thermal state:

\[ \rho = e^{-\beta H} \]

For \( \beta \to \infty \), obtain vacuum state.

\[ \langle \phi_0 | e^{-\beta H} | \phi_1 \rangle = \phi_1 \]

\( \phi = \phi_0 \)

\( \phi = \phi_1 \)

\[ \rho_A = \text{tr}_B e^{-\beta H} \]

Reduced state of \( A \subseteq \Sigma \):

path integral on \([0, \beta] \times \Sigma\)

path integral on plane with half-slit
Rindler decomposition

Rindler wedges correspond to $A = [0,\infty)$ and $B = (-\infty,0]$.

Lorentz boost generator $K$ acts by rotations in Euclidean signature

Similarly, Schmidt decomposition:

$$|\Omega_{AB}\rangle = \sum_i e^{-\pi\omega_i} |i'\rangle|i\rangle$$

Amusing: If $|\Omega_{AB}\rangle$ were product $\Rightarrow$ “firewall” between $A:B$. 

Homework!
Using the **replica trick**, it is easy to compute Renyi entropies:

\[
S_n(\rho) = \frac{1}{1-n} \log \frac{\text{tr}[\rho^n]}{\text{tr}[\rho]^n} = \frac{1}{1-n} (\log Z_n - n \log Z_1)
\]

where \( Z_n = \text{tr}[\rho^n] = \text{tr}[\rho \otimes_n C_n] \) is calculated by the following path integral:
Entanglement entropy for single interval

Can be explicitly computed for spherical regions in conformal field theory.

Cardy–Calabrese: In 1+1d CFT with central charge $c$,

$$S_n = \frac{c}{6} \left( 1 + \frac{1}{n} \right) \log \frac{L}{\epsilon}$$

$$S = \frac{c}{3} \log \frac{L}{\epsilon}$$

Homework: Prove this.

$M_n$ is topologically sphere, compute $Z_n$ from Weyl anomaly.

Alternatively, via 2-point function of twist operators in orbifold CFT:

$$Z_n = \langle \sigma_+ (z_1) \sigma_- (z_2) \rangle_{\text{CFT}^n / Z_n}$$
**Application: c-theorem**

Can use entanglement entropy to construct RG monotone and re-prove \(c\)-theorem.

Suppose we deform “UV CFT” by relevant operator. Then:

\[
S(L \ll \xi) = \frac{c_{UV}}{3} \log \frac{L}{\xi} \\
S(L \gg \xi) = \frac{c_{IR}}{3} \log \frac{L}{\xi'}
\]

**Claim:** \(c(L) = 3 \frac{L}{dS/dL} \) interpolates \(c_{UV}, c_{IR}\) and decreases with \(L\).

**Key idea:** Use strong subadditivity \(S(AB) + S(BC) \geq S(ABC) + S(B)\).

Here:

\[
S(x) + S(y) \geq S(L') + S(L)
\]

Choose \(L' = L + \delta \Rightarrow \frac{d^2...}{d\delta^2} = -\frac{1}{2} L \frac{dS}{dL} \geq 0\)
5. Entanglement in Holography

Black holes and quantum information

Black holes have a thermodynamic temperature and entropy. This entropy is proportional to the area of the event horizon:

\[ S_{BH} = \frac{\text{Area}}{4G} \]

Surprising! Further puzzles arise when we try to quantize: Hawking radiation, information paradox(es), ...

A theory of quantum gravity ought to give microscopic explanations.
Holographic principle and practice

**Holographic principle**: Can all information in a region of space be represented as “hologram” living on boundary?

Susskind, ’t Hooft

**AdS/CFT duality**: Realization in Anti-de Sitter space

Maldacena

Controlled setup to study quantum gravity; including black holes, wormholes, ...

*What can we learn by applying the QI toolkit?*
AdS/CFT Dictionary

CFT

(string) gravity theory

Symmetries ✓

Partition functions:

\[ Z_{\text{CFT}} = Z_{\text{string}} \]

“Extrapolate dictionary”:

\[ O(X) = \lim_{r \to \infty} r^A \phi(r, X) \]

\( \frac{c}{3} = \frac{\ell}{2G_N} \)

- can compute CFT correlation functions:

\[ \int D\phi e^{iS_{\text{eff}}} O_1 \cdots O_n = \langle O_1 \cdots O_n \rangle_{\text{CFT}} \]

What is the bulk dual of entanglement entropy?
Ryu-Takayanagi formula

Ryu-Takayanagi (RT): For static space-times, boundary entropies are computed by area of bulk minimal surface homologous to $A$:

$$S(A) = \min \frac{|\gamma_A|}{4G} + \ldots$$

Entanglement $\Leftrightarrow$ Geometry
Example: AdS$_3$

CFT vacuum state $|\Omega\rangle$ is dual to AdS$_3$ bulk:

Pure state: $S(\Sigma) = 0, \; S(A) = S(A^c) \; \checkmark$

For an interval of length L, recover Cardy formula:

Poincaré coordinates

$$ds^2 = \ell^2/z^2 (dx^2 + dz^2 - dt^2)$$

$$|A_A| = 2\ell \log(L/\varepsilon)$$

$$S(L) = c/3 \log(L/\varepsilon) \; \checkmark$$

Homework: Verify this.
Example: Multiple subsystems

Two boundary subsystems:

\[ S(AB) = S(A) + S(B) \]
\[ I(A:B) = 0 \]

“uncorrelated phase”

\[ S(AB) < S(A) + S(B) \]
\[ I(A:B) > 0 \]

“correlated phase”
**Example: BTZ black hole**

BTZ\(_3\) black hole solution is dual to CFT\(_2\) thermal state \(\rho_\beta\):

\[
\text{Mixed state: } S(\rho_\beta) = \frac{\text{horizon area}}{4G_N} > 0 \quad \checkmark
\]

**Phase transition in entanglement entropy:**

Entanglement shadow: minimal geodesics don’t reach all the way to \(r_+\).
Example: Thermofield double

\[ |\text{TFD}_\beta\rangle = \frac{1}{Z} \sum_E e^{-\beta E/2} |E\rangle |E'\rangle \]

Thermofield double state is purification of thermal state to two CFTs. Bulk dual: Two-sided black hole in static asymptotic AdS space-time.

contains Einstein-Rosen (ER) bridge connecting asymptotic AdS regions

“ER = EPR” Susskind-Maldacena

cf. Rindler wedge analysis
Why should Ryu-Takayanagi hold?

Intuitive generalization of Bekenstein-Hawking formula.

Matches CFT calculations. ✓

Proved under plausible assumptions. ✓ Lewkowycz-Maldacena

Satisfies many nontrivial consistency checks. For example, easy to verify strong subadditivity:

\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]

However, we can prove “too much”…
Holographic entropy laws

Ryu-Takayanagi formula satisfies non-standard entropy inequalities. These are constraints for CFTs to have a gravity dual!

"Monogamy" inequality: \( I(A:B) + I(A:C) \leq I(A:BC) \)

Does not hold general states – not even for all probability distributions. Correlations are not monogamous!

\( \sum_{n} e^{-\beta E_{n}/2} |n\rangle |n\rangle |n\rangle |n\rangle \neq \)

\( \)

\( \rightarrow \) excludes plausible states such as

\( \rightarrow \) can be used to witness multipartite correlations

"=" for bipartite correlated states \( \rho_{AB} \otimes \rho_{AC} \otimes \rho_{BC} \)
How to prove holographic entropy inequalities?

\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]

General method that abstracts inclusion/exclusion reasoning:

“Homology regions” for LHS minimal surfaces partition bulk into \(2^{\text{LHS}}\) regions. \(\Rightarrow\) Hypercube:

- vertices = bulk regions
- edges = surfaces between regions

\(\Rightarrow\) use subsets of hypercube to define homology regions for RHS surfaces

not necessarily minimal

If each edge cut at most once: Entropy inequality is valid!

Homework:
Work out details.
Hypercube proof of monogamy inequality

To illustrate the method, let us prove the “monogamy inequality”, which expands to:

\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

Infinitely many holographic entropy inequalities can so be proved. How to organize systematically?
Holographic entropy cones

For fixed number of subsystems, consider all possible entropy vectors:

\[ C_n = \{(S_{RT}(A_1), ..., S_{RT}(A_1A_2...A_n))\} \]

arbitrary geometries allowed!

This is a polyhedral convex cone – the holographic entropy cone.

faces: entropy inequalities such as \( S(A) + S(B) \geq S(AB) \)

rays: entropy vectors that cannot be written as mixture of others. represented by “extremal geometries”.

can these be identified with microscopic building blocks??
Constraints from entropy inequalities

Can also go the other way and exploit known entropy inequalities to derive gravitational constraints. E.g., using relative entropy:

\[ S(\rho || \sigma) = \text{tr} \rho \log \rho - \text{tr} \rho \log \sigma \geq 0 \]

Perturb around vacuum state:

1st order: linearized Einstein equations  
2nd order: positive energy inequalities

E.g. \[ \int T_{00} \sqrt{g} \geq 0 \]

Much more to be said about holographic entropies (monotonicity of relative entropy, Freedman–Headrick bit threads, ...)
Generalizations

Entropy of bulk fields in region enclosed by RT surface contribute $O(1)$ corrections to entropy:

$$S(A) = \frac{|\gamma_A|}{4G} + S(a)$$

better: minimize joint expression ("generalized entropy")

RT holds in static situations (more generally, in time-reflection symmetric situations).
In general, consider extremal area codimension-2 spacelike bulk surfaces.
Hubeny-Rangamani-Takayanagi (HRT)

Equivalently, Wall’s maximin procedure:
$$S(A) = \max_{\Sigma} \min_{\gamma_A} \frac{|\gamma_A|}{4G}$$
6. Toy Models of Holography

Holography is mysterious...

1) “Extrapolate” dictionary: \[ r^A \phi(X,r) \xrightarrow{r \to \infty} O(X) \]

A puzzle: \[ [\phi(y), O(X)] = 0 \]

2) Ryu-Takayanagi with bulk corrections:

\[ S(A) = \min \frac{|\gamma_A|}{4G} + S(a) \]

3) Bulk reconstruction problem: Every bulk operator should be dual to some boundary operator.

\[ \phi(x) \overset{!?}{=} \int O(X)K(X|x)dX \]

Why do we care? Extrapolate dictionary insufficient if want to study processes behind horizons, understand bulk locality.
Subregion duality:

Can write any bulk operator in $a$ as boundary operator in $A$!

Proved using QI tools. $\checkmark$ Dong–Harlow–Wall, Cotler–…–W

Not known how to do explicitly in most tantalizing situations:

Only when $A = \text{everything}$ or $\phi(x)$ in (smaller) causal wedge of $A$.

$\Rightarrow$ Hamilton–Kabat–Lifschytz–Lowe, Banks et al, Heemskerk et al, …, Harlow TASI
Holography is mysterious

Subregion duality leads to another puzzle:

\[ \phi = O_{AB} = O_{AC} = O_{BC} \]

no common support \( \notin \)
\( AB \cap AC \cap BC = \emptyset \)

Resolution: Only "few" states correspond to any particular semiclassical bulk description.

"[\phi(y), O(X)] = 0" or "O = \phi" only hold (make sense!) on small subspaces of CFT Hilbert space, known as "code subspaces"

Plan: Discuss toy models that reproduce 1)-3) and resolve puzzles by simple QI mechanisms.
Three-Qutrit code

\[ \mathbb{C}^3 \rightarrow \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \]

\[ V|i\rangle = |\tilde{i}\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j+i, j-i\rangle \]

encodes 3-dim in 27-dim space

states \( \rho \) are encoded by \( \tilde{\rho}_{ABC} = V\rho V^\dagger \)

operators \( \phi \) are encoded by \( \tilde{\phi}_{ABC} = V\phi V^\dagger \)

Key fact:

\[ V|i\rangle = (I_A \otimes U_{BC})(|\Phi^+_A \rangle \otimes |i_C\rangle) \]

where \( U_{BC}|j,i\rangle = |j+i,j-i\rangle \)
Three-Qutrit code

This has remarkable consequences:

Ryu-Takayanagi: \( S(A) = \log(3) \)
\( S(AB) = \log(3) + S(\rho) \)
\( S(B) = S(C) \)
\( S(AC) = S(BC) \)

Subregion duality: can decode \( \rho \) from BC alone!
likewise from AB, AC

Heisenberg picture:
\( O_{BC} = U_{BC}(I \otimes \Phi)U_{BC}^\dagger \)
\( \Rightarrow O_{BC}V = V\Phi, \ O_{BC}^\dagger V = V\Phi \)
“\( O_{BC} = \Phi \)”

\( \Rightarrow \text{resolves second puzzle!} \)
Three-Qutrit code

Similarly, if $\phi$ is \textit{any} bulk and $O_A$ \textit{any} boundary operator on $A$:

$$\langle \tilde{i} | [O_A, \tilde{\Phi}_{ABC}] | \tilde{j} \rangle = \langle \tilde{i} | [O_A, O_{BC}] | \tilde{j} \rangle = 0 \quad \text{"}[\phi(x), O(Y)] = 0\text{"}$$

→ resolves \textit{first puzzle}!

\textbf{Quantum error correction} plays important role in recent research in holography (emergence of bulk locality, black hole information paradox, ...)

Verlinde$^2$, Almheiri-Dong-Harlow, ...
7. Error Correction, Decoupling, and Black Holes
Recall: Quantum channels

Quantum channel: Any combination of unitary evolution, partial traces, adding auxiliary systems.

Equivalently, any map that sends $\rho_{AR}$ state $\Rightarrow$ $\rho_{BR}$ state.

$\rho_{BR} = (T \otimes \text{id})(\rho_{AR})$

completely positive & trace-preserving (CPTP)

Examples:

**Basis measurement:**

$M(\rho) = \sum_x <x|\rho|x> \ |x><x|$

**Depolarizing noise:**

$D_p(\rho) = p\rho + (1-p)I/d$

Homework: Check this.
Tools for quantum channels

Choi state: characterizes channel completely!

\[ \Omega_{A'B} = (\text{id} \otimes T)(\Phi^+_{A'A}) \]

Stinespring extension: Isometry \( V \) such that:

\[ T(\rho) = \text{tr}_E(V\rho V^+) \]

⇒ complementary channel:

\[ T^c(\rho) = \text{tr}_B(V\rho V^+) \]

what leaks to environment!

Together: Solve channel problems by (pure) state reasoning!
Example: Basis Measurement

\[ \Omega_{A'B} = (\text{id} \otimes M)(|\Phi_{AA}^+><\Phi_{AA}^+|) = 1/d \sum_{x,y} (\text{id} \otimes M)(|xx><yy|) \]
\[ = 1/d \sum_{x,y} |xx><yy| \otimes M(|xx><yy|) = 1/d \sum_{x} |xx><xx| \otimes |xx><xx| \]
\[ = 1/d \sum_{x} |xx><xx| \]

\[ V|x> = |xx> \]
\[ \text{tr}_E(V|x><y|V^+) = \text{tr}_E(|xx><yy|) = \delta_{xy} |xx><xx| = M(|xx><y|) \]

\[ \Rightarrow \text{Complementary channel: } M^c = M! \]

Homework: Compute Choi + Stinespring for other examples.
Quantum error correction

When building quantum computers, we want to protect against errors (imperfections, noise, decoherence, ...).

To achieve this, redundantly encode “logical” into “physical” qubits:

For example, 3-qutrit code corrects again erasure of any 1 qutrit.

Holography:

Questions:

1) When can we in principle correct?
2) How to correct in practice?
Decoupling criterion

The question: Given a channel $T_{A \rightarrow B}$, when can we reverse it?

Decoupling criterion: Can reverse $T_{A \rightarrow B}$ if and only if the complementary channel $T^c_{A \rightarrow E}$ is constant.

→ exactly what we found for 3-qutrit code
→ very strong form of “no cloning” statement

If reversible: There exists state $|\chi\rangle$ and isometry $W$ such that:

$$\Omega_{AE} = \Omega_{A'} \otimes \chi_E$$

$$I(A':E) = 0$$

or $$|\Omega_{A'AEF}\rangle = |\Phi^+_{AA}\rangle \otimes |\chi_{EF}\rangle$$

Homework: Prove this.
Teleportation revisited

It is instructive to revisit teleportation from this perspective. Consider channel which performs Bell measurement on $\rho_M \otimes I_A/2$:

This is a constant channel since all outcomes are equally likely. By the decoupling criterion, can decode from complementary channel!

First, compute Stinespring extension:

$$U |\Phi^{(xz)}\rangle = |xzxxz\rangle$$

Thus, complementary channel looks like teleportation w/o correction:
Decoupling Inequality

In information theory, random codes are often almost optimal.

\[ V = \begin{array}{c}
A \\
\downarrow \\
B \\
\downarrow \\
\mathbf{U} \\
\downarrow \\
E
\end{array} \]

random unitary

When can we decode A from B?

\[ |\Omega_{A'B'E} \rangle = \begin{array}{c}
A' \\
\downarrow \\
B \\
\downarrow \\
\mathbf{U} \\
\downarrow \\
E
\end{array} \]

Need \( \Omega_{A'E} \approx \Omega_{A'} \otimes \Omega_E \)

The following result addresses these kind of problems:

Decoupling Inequality: Let \( \rho_{ABE} \) state, \( U_{BE} \) random. Then:

\[
\int dU_{BE} \left\| \text{tr}_B (U_{BE} \rho_{ABE} U_{BE}^\dagger) - \rho_A \otimes I_E \right\|_1^2 \leq \frac{d_{AE}}{d_B} 2^{-S_2(\rho)}
\]
Hayden–Preskill protocol

We again model an evaporating black hole by random unitary. After Page time, assume black hole maximally entangled with old radiation.

Now suppose Alice throws her diary into black hole.

How much further radiation do we need to collect so that we can recover diary?

That is, when can we decode A from RR’?

\[ |\Omega_{A'BRR'}\rangle = U |A'BRR'\rangle \]

Need \( \Omega_{A'B} \approx \Omega_{A'} \otimes \Omega_{B} \! \)!

Answer: \( d_A \ll d_R \)

Little more than size of diary - independent of size of black hole. Black hole after Page time is like a mirror, information comes right out.

Homework: Show this using the decoupling theorem with \( B \rightarrow R, E \rightarrow B \).
Holographic teleportation

Wormhole in thermofield double state can be made traversable by weak local classical coupling between the two CFTs.

This holographic teleportation protocol is remarkable: “self-decoding” even though CFT time evolution scrambling!

Recent work constructed toy models using e.g. random unitaries and proposed general QI mechanisms.

Effective interaction only depends on operator sizes.
**Bonus: Relative entropy**

\[
\mathcal{D}(\rho \| \sigma) = \text{tr} \, \rho \left( \log \rho - \log \sigma \right)
\]

\[
\geq 0 \quad \text{iff} \quad \rho = \sigma
\]

well-defined in QFT

\[\Rightarrow S(\rho) = \log d - \mathcal{D}(\rho \| I/d), \quad I(A:B) = \mathcal{D}(\rho_{AB} \| \rho_A \otimes \rho_B), \quad \ldots\]

**Pinkser inequality:**

\[
\mathcal{D}(\rho \| \sigma) \geq \frac{1}{2\ln 2} \| \rho - \sigma \|_1^2
\]

**Data processing inequality:**

\[
\mathcal{D}(\rho \| \sigma) \geq \mathcal{D}(T(\rho) \| T(\sigma)) \quad \Rightarrow \text{strong subadditivity}
\]

“=“ \(\Leftrightarrow\) can reverse channel on pair of states \(\rho, \sigma\)

How so? Use Petz map:

\[
\mathcal{D}(\rho) = \sigma^{1/2} T^\dagger(T(\sigma)^{-1/2} \rho \ T(\sigma)^{-1/2}) \ \sigma^{1/2}
\]
Back to our toy model

Toy model has no geometry – just a single site!

To go beyond, let’s focus on the RT formula:

$$V|i\rangle = \tilde|i\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j + i, j - i\rangle$$

$$\tilde{0} = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j, j\rangle$$

Can we glue together many such states (or codes)?
8. Tensor Network Toy Models

Many-body quantum states

Many-body quantum states have exponentially large description

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle \]

tensor with n indices

In practice: entanglement is local, correlations decay rapidly

→ can hope for more efficient description:

Key idea: start with entangled pairs...

...and apply local transformations:

e.g. ‘cat’ state \( |0\ldots00\rangle + |1\ldots11\rangle \) from \( |00\rangle \rightarrow |0\rangle, \ |11\rangle \rightarrow |1\rangle \)
Tensor networks as a tool

**Tensor network**: state defined by contracting network of (local) tensors

\[ |\psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

- **MPS**: …
  - White, Fannes-Nachtergaele-Werner, Östlund-Rommer
- **PEPS**: …
  - Verstraete-Cirac

**Numerical tool**: efficient **variational classes**

- Provably so for gapped theories in 1+1d (Hastings)
- Can even have interpretation as **quantum circuits**

**Powerful theoretical formalism**, provides “dual” descriptions of complex phenomena → quantum phases, topological order, …
Computing with tensor networks

Very similar to path integral reasoning:

\[ \langle \Psi_1 \Psi_2 \rangle = \text{network} = \cdots \]

\[ \langle \Psi_1 \Omega_0 \Omega_2 \Psi_2 \rangle = \text{network} = \cdots \]

Can formally obtain tensor networks by trotterizing \( e^{-\beta H} \).

What is the role of the network geometry?
Entrophy in tensor networks

Entanglement entropy satisfies “Ryu-Takayanagi bound”:

\[
S(A) \leq N |\gamma_A|
\]

where
\[
\gamma_A = \text{minimal cut}
\]

In general, the bound is not saturated…

Tantalizing: Picture shows Vidal’s MERA tensor network. Used for critical theories, it looks like a time slice of AdS!  

Swingle
Why does the bound hold?

Thus, the Schmidt rank is at most $2^N |\gamma_A|$. 

Thus, the bound saturated if $L$, $R$ are unitaries (or isometries)!

NB: Bound saturated if $L$, $R$ are unitaries (or isometries)!
Holography from tensor networks

Want “exactly solvable” toy models of holographic duality:

Approach: Define boundary state via tensor network in bulk

- simple bulk tensors, e.g. random and large $N$
- emergent Ryu-Takayanagi law!

$S(A) \approx N |\gamma_A|$

Mostly works in any geometry. By now, many variations known.
Assume each local tensor is perfect = isometry in all possible directions.

#in ≤ #out exist! e.g. 3-qutrit code, 5-qubit code, ...

Choose orientations such that $\gamma_A \rightarrow A, A^c$
Then: $V, W$ isometries and RT formula holds

Always possible for graphs with “negative curvature” and A “single interval”.

Concrete and intuitive!

How to generalize?
Random tensor model

Choose random bulk tensors of large bond dimension.

→ emergent RT formula

Three interpretations:

1. Random tensors \(\approx\) perfect
2. Entanglement distillation protocol
3. Disorder average \(\rightarrow\) ferromagnetic spin model
   large \(N\) \(\rightarrow\) low \(T\)

Remarkably, behavior matches fixed-area states in holography.

Dong-Harlow-Marolf
Derivation of Ryu–Takayanagi law

Setup:

Arbitrary lattice or graph. Tensors are chosen i.i.d. from Haar measure.

Recall: In any tensor network: $S(A) \leq N |\delta_A|$.

Strategy: Lower bound $S_2(A)$ using replica trick.
Calculation of Renyi entropy

\[ S_2(A) = -\log \text{tr} \rho_A^2 \]

\[ |\Psi\rangle = \left( \bigotimes_{\langle x, y \rangle} \langle x y | \right) \left( \bigotimes_x |V_x\rangle \right) \]

Replica trick: \([\text{tr} \rho_A^2 = \text{tr}(\rho \otimes \rho)F_A] \)

97/124
Calculation of Renyi entropy

$$S_2(A) = -\log \text{tr} \rho_A^2$$

$$\Psi = \left( \bigotimes_{\langle x,y \rangle} \langle xy \rangle \right) \left( \bigotimes_x |V_x\rangle \right)$$

Replica trick: $$\text{tr} \rho_A^2 = \text{tr} (\rho \otimes \rho) F_A$$
Calculation of Renyi entropy

\[ S_2(A) = - \log \text{tr} \rho_A^2 \]

\[ |\Psi\rangle = \left( \bigotimes_{\langle x,y \rangle} \langle xy | \right) \left( \bigotimes_x |V_x\rangle \right) \]

Replica trick: \[ \text{tr} \rho_A^2 = \text{tr} [\rho \bigotimes \rho] F_A \]
Calculation of Renyi entropy

\[ S_2(A) = - \log \text{tr} \rho_A^2 \]

\[ |\Psi\rangle = \left( \bigotimes_{\langle x,y \rangle} |xy \rangle \right) \left( \bigotimes_x |V_x \rangle \right) \]

**Replica trick:**
\[ \text{tr} \rho_A^2 = \text{tr} \left( \rho \bigotimes \rho \right)^F \]

\[ A \]

100/124
Calculation of Renyi entropy

\[ S_2(A) = -\log \text{tr} \rho_A^2 \]

\[ |\Psi\rangle = \left( \bigotimes_{x,y} \langle xy | \right) \left( \bigotimes_x |V_x\rangle \right) \]

**Replica trick:**
\[ \text{tr} \rho_A^2 = \text{tr}(\rho \otimes \rho)F_A \]

\[ \text{tr} \rho_A^2 = \]

\[ A \]

\[ \rho \]

\[ \Lambda \]
Calculation of Renyi entropy

\[ S_2(A) = - \log \text{tr} \rho_A^2 \]

\[ |\Psi\rangle = \left( \bigotimes_{(x,y)} \langle x | y \rangle \bigotimes_{x} |V_x\rangle \right) \]

**Replica trick:** \[ \text{tr} \rho_A^2 = \text{tr}(\rho \otimes \rho)F_A \]

\[ \text{tr} \rho_A^2 = \]

\[ |V_x\rangle \langle V_x| \otimes^2 = \]
Calculation of Renyi entropy

\[ S_2(A) = -\log \text{tr} \rho_A^2 \]

\[ |\Psi⟩ = \left( \bigotimes_{x,y} ⟨x,y| \right) \left( \bigotimes_x |V_x⟩ \right) \]

Replica trick: \[ \text{tr} \rho_A^2 = \text{tr}(\rho \otimes \rho)F_A \]

\[ \text{tr} \rho_A^2 = \frac{1}{|V_x⟩⟨V_x|^2} \propto = I + F \]

(recall from Page curve calculation!)
Calculation of Renyi entropy

\[ S_2(A) = - \log \text{tr} \rho_A^2 \]

Replica trick:

\[ \text{tr} \rho_A^2 = \text{tr}(\rho \otimes \rho) F_A \]

Each loop is trace: factor D
Calculation of Renyi entropy

\[ S_2(A) = -\log \text{tr} \rho^2_A \]

\[ |\Psi\rangle = \left( \bigotimes_{\langle x, y \rangle} |x y\rangle \right) \left( \bigotimes_x |V_x\rangle \right) \]

Replica trick:

\[ \text{tr} \rho^2_A = \text{tr} (\rho \otimes \rho) F_A \]

\[
\text{tr} \rho^2_A = \frac{1}{\text{tr} \rho^2_A} \propto
\]

Ising variables & boundary conditions!
Result: Renyi entropy and Ising model

\[ S_2(A) = -\log \text{tr} \rho_A^2 \]

\[ \text{tr} \rho_A^2 \simeq Z_A = \sum_{\{s_x\}} e^{-\log D \times \frac{1}{2} \sum_{\langle x, y \rangle} (1 - s_x s_y)} \]

\[ S_2(A) \simeq -\log Z_A \]

free energy

\[ \frac{1}{T} \text{ ferromagnetic Ising} \]

\[ \text{... + O}(1) \text{ if multiple minimal domain walls. D}_{\text{crit}} \text{ from Ising physics.} \]

Homework: Verify this.
Result: Renyi entropy and Ising model

$$S_2(A) = -\log \text{tr} \rho_A^2$$

$$\text{tr} \rho_A^2 \simeq Z_A = \sum_{\{s_x\}} e^{-\frac{1}{2} \sum_{(x,y)} (1 - s_x s_y) + \log D}$$

$$S_2(A) \simeq -\log Z_A \simeq \log D |\gamma_A|$$ \hspace{1cm} \text{large } D \text{ / low } T \hspace{1cm} \text{free energy dominated by minimal energy configuration = domain wall}

→ Ryu-Takayangi law for the entanglement entropy:

$$S(A) \simeq \log D |\gamma_A|$$

Homework: Verify this.

... + O(1) if multiple minimal domain walls. $D_{\text{crit}}$ from Ising physics.
Discussion

Higher Renyi entropies (and fluctuations) controlled by $S_n$ spin model.

$S(A) \approx S_2(A) \Rightarrow \text{Flat entanglement spectrum.}$

if RT surface unique

Similar to proof of RT formula. But in AdS/CFT, replicas need to satisfy Einstein equations $\Rightarrow$ nontrivial entanglement spectrum

How to incorporate into toy models?

Special case of general mechanism:

$|\Psi\rangle = \langle \Omega | (\otimes_x |V_x\rangle) \Rightarrow \text{"}S_2(A) \approx \min_a S_2(a)\text{"}$
Multiparty entanglement distillation: create entanglement between Alice & Bob with help of Charlies by measurements & classical communication.

General mechanism for producing Ryu-Takayanagi from area law state!
AdS/CFT is duality between two theories = “dictionary” that maps states & observables. How to incorporate into toy model?

Approach: Define bulk-boundary mapping via tensor network

= combination of both toy models

red legs: bulk degrees
black legs: boundary degrees

“logical” bulk states are encoded in “physical” boundary Hilbert space

Toy model of how bulk quantum fields get encoded in boundary CFT.
Holographic codes

Reproduce key QI features of AdS/CFT correspondence:

✓ subregion duality

Q. information deep in bulk is better protected.

✓ bulk corrections to entanglement entropy:

\[ S(A) \approx \min \{ N |\gamma_A| + S(a) \} \]

entanglement vs geometry:

Maldacena–Susskind, Verlinde, ...

adding too many states “breaks” code and creates entanglement shadow (≈ horizon)

cf. BH microstates
Subregion duality

Suppose that we can show...

\[ U \times V \]

This would imply both subregion duality and the Ryu-Takayanagi formula with bulk correction!
Proof of subregion duality

1) HaPPY argument: Choose orientations s.th. $a\gamma_A \rightarrow A$, $b\gamma_A \rightarrow B$.

Interpretation: Holographic codes are macroscopic erasure codes built from microscopic ones (perfect tensors).

2) Decoupling argument: Only need to prove that $I(a':b'B) = 0$. Why? See later!

⌘ Can prove geometrically since Choi state satisfies Ryu-Takayanagi!
Mutual information calculation

Schematically:

\[ S(a) = \log(d) |a| \]
\[ S(b_B) = \log(D) |\gamma_A| \]
\[ S(ab_B) = \log(D) |\gamma_A| + \log(d) |a| \]
\[ \Rightarrow I(a:bB) = 0 \]

Assume bulk legs have small dimension \( d \ll D \).
9. Subregion Duality and Subsystem Error Correction

Literature: https://arxiv.org/abs/1607.03901
Subregion duality

Let us talk more systematically about the quantum information structure of subregion duality. Consider an isometry:

\[ \rho_{ab} \text{ state } \Rightarrow \rho_{AB} = V \rho_{ab} V^\dagger \]

Subsystem error correction: When can we recover \( a \) from \( A \)?

More subtle than what we discussed last lecture. There we had no “b” system - now \( \rho_{ab} \) can be correlated or entangled!
Subsystem error correction

The following conditions are equivalent:

1) There is a channel $D_{A \to a}$ such that:
   $$D(\rho_A) = \rho_a \text{ for all } \rho_{ab}$$

2) For all $\phi_a$, exists $O_A$ such that:
   $$V\phi_a = O_A V \text{ and } V\phi_a^\dagger = O_A^{\dagger} V$$

3) For all $\phi_a$, and $XB$:
   $$[\phi_a, V^\dagger XB V] = 0$$

4) Decoupling:
   $$I(a':b'B) = 0$$
   i.e.
   $$\Omega_{A'b'B} = \Omega_{A'} \otimes \Omega_{b'B}$$

Aside: 2) allows computing correlation functions – even if we use different subsystems for each operator:

$$\langle \phi_{ab} \phi'_{bc} \rangle = \langle O_{AB} O'_{BC} \rangle$$
Proof sketch of equivalence

1) There is a channel $D_{A \rightarrow a}$ such that:

$D(\rho_A) = \rho_a$ for all $\rho_{ab}$

$\Leftrightarrow$ Stinespring extension:

$O_A = W^\dagger \phi_a W$

$V \phi_a = O_A V$

$V \phi_a^+ = O_A^+ V$

$\Leftrightarrow$ Choi state:

$[\phi_a, V^\dagger X_B V] = 0$

$I(a':b'B) = 0$

compare purifications
Complementary recovery

When can we recover $a$ from $A$ and $b$ from $B$? Result:

Normal form:

Ryu-Takayanagi formula:

\[ S(A) = c + S(a) \text{ for all } \rho_{ab} \]
\[ S(B) = c + S(b) \text{ for all } \rho_{ab} \]

The punchline: RT formula is also sufficient for subregion duality.

"proof" that latter holds in AdS/CFT
**Bonus: Proof that Ryu-Takayanagi implies complementary recovery**

Assume: \( S(A) = c + S(a) \) for all \( \rho_{ab} \)

Use 1st law: \( \text{tr}[K_a \delta \rho_{ab}] = \delta S(a) = \delta S(A) = \text{tr}[K_A \delta \rho_{AB}] = \text{tr}[V^t K_A V \delta \rho_{ab}] \)

\[ V^t K_A V = c + K_a \quad \text{for modular Hamiltonians} \quad K_a = -\log \rho_a, \quad K_A = -\log \rho_A \]

\[ D(\rho_A || \sigma_A) = -\text{tr}[\rho_A K_{\rho_A}] - \text{tr}[\rho_A K_{\sigma_A}] = -\text{tr}[\rho_a V^t K_{\rho_A} V] - \text{tr}[\rho_a V^t K_{\sigma_A} V] = \ldots \]

\[ D(\rho_A || \sigma_A) = D(\rho_a || \sigma_a) \quad \text{for all} \quad \rho_{ab} \text{ and } \sigma_{ab} \]

\[
\rho_{ab} = e^{i\phi bs} \sigma_{ab} e^{-i\phi bs}
\]

Use Petz map to obtain decoder \( D_{A \rightarrow a} \)

[\( \Phi_b, V^t X_A V \)] = 0

Homework: Fill in the details.
Decoding the hologram \textit{(using error correction)}

This proof of “entanglement wedge” reconstruction property is \textit{nonconstructive} & \textit{nonrobust}.

\textbf{How to find boundary reconstruction of local bulk operator?}

Banks et al, Hamilton et al, Kabat et al, Heemskerk et al, Lin et al, Faulkner–Lewkowycz, ...

Recall: Only understood in special cases.

Recent progress in quantum error correction led to \textit{robust} proof.

Cotler–…–W, Kitaev–Yoshida, Hayden–Penington
Theorem models situation where \textbf{minimal surface} can be considered \textbf{fixed} for all states in code subspace (no backreaction).

In general, \textbf{state-dependent!} “Quantum” minimal surface obtained by minimizing generalized entropy:

\[
S(A) = \min \left\{ \frac{|\gamma_A|}{4G} + S(a) \right\}
\]

realized in \textbf{random tensor network model}! ✓

This form of subregion duality has featured crucially in very recent research on the \textbf{black hole information paradox} that seeks to give a bulk picture of black hole evaporation.

→ Penington, Almheira et al, lectures by Netta?
Summary

Whirlwind tour through some key concepts and tools of quantum information, motivated by applications to QFT and holography:

States, Channels, Entropy
Entanglement of Pure and Mixed States
Entanglement in Field Theory and Holography
Toy Models of Holography
Quantum Error Correction and Decoupling

No time for quantum computing: circuits, algorithms, complexity, ... 😞

Slides: https://staff.fnwi.uva.nl/m.walter/
The road ahead

Holography predicts remarkable connection between geometry and entanglement

Quantum information offers tools, models, mechanisms from tensor networks to QEC

Ongoing research to exploit connections

Motivation ranges from trying to understand the emergence of space-time from quantum mechanics to learning how dualities can help simulate complex quantum systems on (quantum) computer...

Thank you for your attention!!!
(And thanks Claire, Freek, Jackson, Tarek!!!)