Analytic algorithms for the moment polytope

Cole Franks
Rutgers University
Based on joint work with

Peter Bürgisser  Ankit Garg  Rafael Oliveira

Michael Walter  Avi Wigderson

Mainly from “Towards a theory of non-commutative optimization: geodesic 1st and 2nd order methods for moment maps and polytopes”
FOCS 2019
1. Moment polytopes by example
2. Algorithms for the general problem
Moment polytopes
**Motivating question**

<table>
<thead>
<tr>
<th><strong>Horn’s problem:</strong></th>
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If so, can one find the matrices efficiently?
Let $\mathcal{V} = \mathbb{P}(\text{Mat}(n)^2)$, define

$$\mu : \mathcal{V} \rightarrow \text{Herm}(n)^3$$

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Note $\text{eigs}(AA^\dagger) = \text{eigs}(A^\dagger A)$, so

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Moment polytopes

- $G = \text{GL}(n)$
- $\pi : G \to \mathbb{C}^m$ a representation of $G$ where $U(n)$ acts unitarily
- $\mathcal{V} \subset \mathbb{P}(\mathbb{C}^m)$ a projective variety fixed by $G$

Moment map is the map $\mu : \mathcal{V} \to n \times n \text{ Hermitians} =: \text{Herm}(n)$ given by

$$\mu : \mathcal{V} \mapsto \nabla_{H \in \text{Herm}(n)} \log \| e^H \cdot \mathcal{V} \|$$

$i\mu$ is a moment map for $U(n)$ in the physical sense! In particular:

**Theorem (Kirwan)**

Image of

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Encode asymptotic representation theory of coordinate ring of $\mathcal{V}$!

**Theorem (Mumford, Ness ’84, Brion ’87)**

Let $V_{G,\lambda}$ denote irrep of $G$ of type $\lambda$. Then

$$\bigcup_k \frac{1}{k} \{ \lambda : V_{G,\lambda} \subset \mathbb{C}[\mathcal{V}]_k \} = \Delta(\mathcal{V}) \cap \mathbb{Q}^n!$$

Additional math (Schur-Weyl duality, Saturation [KT00])

Horn polytope $\cap (\mathbb{Z}^n)^3 = \{(\lambda_1, \lambda_2, \lambda_3) : V_{GL(n),\lambda_3} \in V_{GL(n),\lambda_1} \otimes V_{GL(n),\lambda_2}\}$
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## Algorithmic tasks

**Input** $(\mathcal{V}, \pi, \lambda)$

- Projective variety $\mathcal{V}$ as arithmetic circuit parametrizing it
- Representation $\pi$ as its list of irreducible subrepresentations as elements of $\mathbb{Z}^n$
- Target $\lambda \in \mathbb{Q}^n$

1. **membership**: determine whether $\lambda$ in $\Delta(\mathcal{V})$.
2. **$\varepsilon$-search**: given $\lambda \in \mathbb{R}^n$, either find an element $\nu \in \lambda$ such that
   - $\|\mu(\nu) - \text{diag}(\lambda)\| < \varepsilon$, OR
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Algorithm for $\varepsilon$-search for Horn polytope (F18)

Input: $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^n)^3$ and $\varepsilon > 0$.

1. Choose $A_1, A_2$ at random. Define

$$ \mu_1 = A_1 A_1^\dagger, \quad \mu_2 = A_2 A_2^\dagger, \quad \mu_3 = A_1^\dagger A_1 + A_2^\dagger A_2. $$

Want $\mu_i = \text{diag}(\lambda_i)$

2. while $\|\mu_3 - \text{diag}(\lambda_3)\| > \varepsilon$, do:
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1. Is membership in $P$?
   - For tori ($G = \mathbb{C}_X^n$) Folklore, [SV17]
   - For Horn polytope, by saturation conjecture [MNS12]
   - For $\lambda = 0$ for quiver representations [GGOW16, IQS17, BFGOWW19]

2. Is it in $RP$?
   - We think so in general, but no proof yet!

3. Is it in $NP$ or $coNP$?
   - In $NP \cap coNP$ for $\mathcal{V} = \mathbb{P}(\mathbb{C}^m)$ [BCM17]
   - Not known in general!
Complexity of moment polytope membership?

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General algorithms
Convert $\varepsilon$-search to an optimization problem

For $b \in B :=$ upper triangular matrices, define

$$\text{cap}_\lambda(v) := \inf_{b \in B} \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$  

Kempf-Ness Theorem

$$\lambda \in \Delta(\mathcal{V}) \iff \text{cap}_\lambda(v) > 0 \text{ for generic } v \in \mathcal{V}$$

$\varepsilon$-search reduces to finding algorithm for the following:

- Given $b$ with $\|\mu(b \cdot v) - \text{diag}(\lambda)\| > \varepsilon$,
- Output $b'$ with

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$$\frac{\|b' \cdot v\|}{\prod_i |b'_{ii}|^{\lambda_i}} < (1 - \delta) \frac{\|b \cdot v\|}{\prod_i |b_{ii}|^{\lambda_i}}.$$
Optimization algorithms

Alternating minimization: \( \text{poly}(1/\varepsilon) \) time [BFGOWW18]

- Tensor products of easy reps e.g. Horn, \( k \)-tensors

\[ \log \text{cap}_\lambda(v) \text{ can be cast as a geodesically convex program!} \]

Domain is positive-semidefinite matrices; geodesics through \( P \) take the form \( \sqrt{P} e^{Ht} \sqrt{P} \)

Geodesic gradient descent: \( \text{poly}(1/\varepsilon) \) time [BFGOWW19]

- Any representation, e.g. \( V = \wedge^k \mathbb{C}^n, \text{Sym}^k \mathbb{C}^n \), arbitrary quivers

Geodesic trust-regions: \( \text{poly}(\log(1/\varepsilon), \log \kappa) \) time [BFGOWW19]

- \( \kappa \) is smallest condition-number of an \( \varepsilon \)-optimizer for \( \text{cap}_\lambda(v) \)
- Polynomial for some interesting cases, e.g. arbitrary quivers with \( \lambda = 0 \)
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Open problems

1. Is moment polytope membership in $\text{NP} \cap \text{coNP}$, or even $\text{RP}$ or $\text{P}$?

2. Membership is in $\text{P}$ for Horn’s problem. But how about $\exp(-\text{poly})$-search?

3. If $(A_1, A_2)$ a random pair of matrices, does $\text{cap}_\lambda(A_1, A_2)$ have an $\epsilon$-minimizer with condition number at most

$$\exp(\text{poly}(\log(1/\epsilon), \langle \lambda \rangle))$$
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Merci!