Quantum Information

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Seek to leverage laws of QM for information processing...

Quantum Information

communication cryptography

networks algorithms

quantum bits computation complexity

entanglement error correction

tensor networks quantum simulation

…but also toolbox and language for studying q. many-body systems.
Physics vs Information: Thermodynamics

Irreversibility (2\textsuperscript{nd} law) vs coarse graining

Boltzmann, Gibbs, ...

Thermodynamics of computation: Cost of erasing a bit?

\[ W \geq kT \ln(2) \]

Most logic gates are irreversive. Is there a fundamental cost to computing? No!

Bennett (1973): Efficient reversible computing is possible!
Physics vs Information: Computation

Simulating quantum physics difficult for classical computers.
   Hilbert space is exponentially large

Why don’t we build a quantum computer? Feynman, Deutsch, ...

Shor’s algorithm (1984): quantum computers may offer vast speedups for classical problems

\[ N = pq \text{ in time} \quad \text{poly}(\log N) \]

Google “quantum supremacy” experiment (2019)

Today, quantum simulation still one of most promising applications.
Physics vs Information: Language and Toolbox

Quantum information is **different**: No cloning, uncertainty principle, Bell violations, entanglement, decoherence, ...

**QIT** offers **language and toolbox** to study and exploit these phenomena.

Examples:

- Uncertainty principle $\rightarrow$ quantum cryptography
- Bell violations $\rightarrow$ device-independent control
- Entanglement $\rightarrow$ many-body physics

In recent years, exciting research at interface of quantum information with QFT and gravity.
Goal: Discuss language, toolbox, key concepts of quantum information. Survey applications to holography.

Today: States, Channels, Entropy, Entanglement

Tue: Entanglement in Mixed States, Entanglement in QFT

Wed: Entanglement in Holography, Toy Models of Holography

Thu: Decoupling, Black Holes

Fri: Tensor Network Models, Error Correction

Homework and open problems throughout ➔ exercise class by Freek
Interrupt me!

If too slow (or too fast), please let me know. 😊

If not detailed enough, please ask. 😊
1. States, Channels, Entropy

Literature: Lectures Notes “Quantum Information Theory” (https://staff.fnwi.uva.nl/m.walter/qit20/)
Quantum states

Density operators on Hilbert space:

\[ \rho = \sum_{x} p_x |\Psi_x\rangle\langle\Psi_x| \]

Pure states: \( \rho = |\Psi\rangle\langle\Psi| \)

Mixed states model ensembles \( \{p_i, \rho_i\} \):

\[ \rho = \sum_{i} p_i \rho_i \]

States of qubit: Bloch ball

|0⟩₀

|1⟩₀

|0⟩₁

|1⟩₁

Classical
Entropy

\[ S(\rho) = -\text{tr} \rho \log \rho \]

\[ = -\sum_x p_x \log p_x \]

Von Neumann entropy:

- only depends on nonzero eigenvalues: \( S(\rho) = S(U\rho U^\dagger) \)

\[ 0 \leq S(\rho) \leq \log(d) \]

Pure state: \( \rho = I/d \)

Modular Hamiltonian:

\[ K_\rho = -\log \rho \]

State-dependent, often nonlocal

"First law of entanglement"

\[ S(\rho + \delta\rho) = S(\rho) + \text{tr}[\delta\rho K_\rho] + \ldots \]

Proof? Homework!
Renyi entropies and replica trick

Von Neumann entropy often difficult to compute → Renyi entropies:

\[ S_n(\rho) = \frac{1}{1-n} \log \text{tr}[\rho^n] \]

\[ = (1-n)^{-1} \log \sum_x p_x^n \]

\( S_0(\rho) = \log \#\text{nonzero eigenvalues} \)
\( S_1(\rho) = S(\rho) \)
\( S_2(\rho) = -\log \text{tr}[\rho^2] \)

\[ \log(d) \geq S_0(\rho) \geq S(\rho) \geq S_2(\rho) \geq ... \geq 0 \]

equal if \( \rho \) flat spectrum

Easy to calculate for integer \( n>1 \):

\[ \text{tr}[\rho^2] = \text{tr}[\rho \otimes^2 F] \]

where

\[ F |xy> = |yx> \]

swap trick

\[ \text{tr}[\rho^n] = \text{tr}[\rho \otimes^n C_n] \]

where

\[ C_n |x_1x_2...> = |x_2x_3...x_1> \]

Proof? Just expand it.
Joint systems

Reduced states of global states $\rho_{AB}$ are given by partial trace:

$$\rho_A = \text{tr}_B(\rho_{AB})$$

$$\langle a|\rho_A|a'\rangle = \sum_b \langle ab|\rho_{AB}|a'b\rangle \quad \Rightarrow \quad \langle O_A\rangle_{\rho_A} = \langle O_A\rangle_{\rho_{AB}}$$

Maximally entangled state (Bell/EPR pair):

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)$$

$$S_{AB} = \frac{1}{2} \left( |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \right)$$

$$S_A = \frac{1}{2} \left( |00\rangle\langle 00| + |11\rangle\langle 11| \right) = \frac{I}{2} \quad \text{maximally mixed}$$

Thus, pure states often have mixed reduced states. Conversely:

Any state $\rho_A$ has a purification $\rho_{AB} = |\Psi_{AB}\rangle\langle \Psi_{AB}|$. 
Correlations

We say that a state is correlated if not a product:

\[ \rho_{AB} \neq \rho_A \otimes \rho_B \]

\[ \langle O_A O'_B \rangle \neq \langle O_A \rangle \langle O'_B \rangle \]

for some pair of observables

Correlations can have quantum or classical origin:

Maximally entangled state:

\[ |\Phi^+_{AB} \rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \]

Max. classically correlated:

\[ \delta_{AB} = \frac{1}{2} (|00\rangle \langle 00 | + |11\rangle \langle 11 |) \]

In both cases, \( \rho_A = \rho_B = I/2 \), but \( \rho_{AB} \neq I/4 \).

How to quantify correlations?
Mutual information

Mutual information:

\[ I(A:B) = S(A) + S(B) - S(AB) \]

\[ \geq 0 \]

\[ = 0 \text{ iff product} \]

\[ |\langle O_A O'_B \rangle - \langle O_A \rangle \langle O'_B \rangle| \leq \| O_A \| \| O'_B \| \sqrt{2 \ln(2)} \ I(A : B) \]

Strong subadditivity (SSA):

\[ I(A:BC) \geq I(A:B) \]

Fundamental, intuitive, difficult to prove.

\[ |\Phi^+_{AB} \rangle = \frac{1}{\sqrt{d}} \sum_x |xx\rangle \]

\[ \gamma_{AB} = \frac{1}{d} \sum_x |xx\rangle \langle xx| \]
Quantum channels

What are the most general transformation of quantum states?

\[ \rho \rightarrow ??? \rightarrow \rho' \]

**Quantum channel:** Any combination of unitary evolution, partial traces, adding auxiliary systems.

Mathematically: Completely positive trace-preserving maps.

**Data processing inequality:**

\[ I(A:B) \geq I(A':B') \]

...if \( \rho_{A'B'} \) obtained from \( \rho_{AB} \) by quantum channels \( A \rightarrow A', \ B \rightarrow B' \).

Homework: Prove this using SSA.
Application: Holevo bound

How many bits can we communicate by sending 1 qubit?

Sender

$\{0,1\}^n \ni x \rightarrow \text{encoder}$

Receiver

$\rho(x) \rightarrow \text{decoder} \rightarrow y$

1 qubit state

Challenge: Do not know optimal states nor optimal decoder!

$$\rho_{XB} = 2^{-n} \sum_x |x\rangle\langle x| \otimes \rho(x) \quad \rightarrow \quad \rho_{XY} = 2^{-n} \sum_x |xx\rangle\langle xx|$$

...if can decode perfectly. Using the data processing inequality:

$$n = I(X:Y) \leq I(X:B) = H(B) - \sum_x p_x H(\rho(x)) \leq \log 2 = 1$$

Homework: Verify this.

1 bit/qubit $\Rightarrow$ no quantum advantage!

FIX TYPO
2. Entanglement

Literature: Lectures on “Symmetry and Quantum Information”
(https://staff.fnwi.uva.nl/m.walter/qit18/)
We say that a state is **separable** if mixture of product states:

\[
\rho_{AB} = \sum_i p_i \rho^{(i)}_A \otimes \rho^{(i)}_B
\]

Otherwise, the state is called **entangled**.

Motivation: classical correlations ≠ entanglement

Separable states are precisely those that can be created by **Local Operations and Classical Communication** (LOCC).

That is, to create **entanglement** need to send quantum systems.

Homework: Show this.
Entanglement in pure states

For pure states, the situation simplifies.

$|\Psi_{AB}\rangle$ is entangled if not a product:

$|\Psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\phi_B\rangle$

That is, all correlations in pure states boil down to entanglement.

[Headrick]
**Schmidt decomposition**

\[ |\Psi_{AB}\rangle = \sum_{i=1}^{r} s_i |e_i\rangle \otimes |f_i\rangle \]

- Schmidt rank
- Orthogonal
- Schmidt coefficients, >0

\[ \rho_A = \sum_{i=1}^{r} s_i^2 |e_i\rangle \langle e_i| \]

\[ \rho_B = \sum_{i=1}^{r} s_i^2 |f_i\rangle \langle f_i| \]

- Reduced states have same eigenvalues, entropies, ... and characterize entanglement:

\[ |\Psi_{AB}\rangle \text{ product} \iff r = 1 \iff \rho_A \text{ pure} \iff \rho_B \text{ pure} \]

- Any two purifications of \(\rho_A\) are related by isometry on B
Extensions and Monogamy

Even if $\rho_{AB}$ mixed: \[ \rho_A \text{ pure } \Rightarrow \rho_{AB} = \rho_A \otimes \rho_B \]

Take purification $|\Psi_{ABC}\rangle$ of $\rho_{AB}$. Since $\rho_A$ pure, $|\Psi_{ABC}\rangle = |\Psi_A\rangle \otimes |\Psi_{BC}\rangle$.

This implies that pure state entanglement is monogamous:

In contrast, classical correlations can be arbitrarily shared.
Entanglement entropy

Schmidt decomposition suggests to quantify entanglement by the entropy of reduced states ➔ Entanglement entropy:

\[ 0 \leq S_E = S(A) = S(B) \leq \log d_A \leq \log d_B \]

- product state
- maximally entangled

**Interpretation:** Optimal conversion rate with Bell pairs:

\[
|\Psi_{AB}\rangle \otimes^n \xleftrightarrow{\text{LOCC}} (|00\rangle + |11\rangle) \otimes S_{E\text{n}}
\]

\( n \to \infty \) copies error \( \to 0 \)

- entanglement transformations “reversible”
- Bell pairs = unit of entanglement

\( n \to \infty \) copies error \( \to 0 \)

\{ for pure states \}
Suppose a black hole is created from infalling matter and we watch it evaporate.

\[ R = \text{Hawking radiation emitted up to some time} \]
\[ B = \text{black hole = later Hawking radiation} \]

A semiclassical calculation suggests entropy of radiation increases until the end. But in a unitary theory, radiation will be pure once BH has evaporated...

Intuitively, early radiation is entangled with black hole, while late radiation is entangled with early radiation.
Simplest toy model: Assume that evaporation described by random unitary evolution.

\[ |\Psi_{BR}\rangle = \text{random pure state} \]

\[ b = \log d_B \]
\[ r = \log d_R \]

Page's theorem: For typical states,

\[ S_E = \min(b,r) - O(1) \]

almost maximal!

It would be more physical to consider a random state in a fixed total energy subspace or a random Hamiltonian evolution.
Derivation of the Page formula

Idea: Lower-bound average Renyi-2 entropy $S_2(R)$ using swap trick.

Key formula: \[ \psi^{\otimes 2} = \frac{I + F}{d(d + 1)} \] for random $\psi = |\psi\rangle \langle \psi|$

Apply this to $|\psi\rangle = |\psi_{BR}\rangle$:

\[ \psi_{BR}^{\otimes 2} = \frac{I_{BB} \otimes I_{RR} + F_{BB} \otimes F_{RR}}{d_B d_R (d_B d_R + 1)} \]

\[ \text{tr} \psi_R^2 = \text{tr} \psi_{BR}^{\otimes 2} F_{RR} \leq \text{tr} \left( I_{BB} \otimes F_{RR} + F_{BB} \otimes I_{RR} \right) = \frac{1}{d_R} + \frac{1}{d_B} \]

\[ S_2(R) \geq -\log \text{tr} \psi_R^2 \geq -\log \left( \frac{1}{d_R} + \frac{1}{d_B} \right) \geq \min(b, r) - 1 \]

Homework: Verify this.

Jensen’s inequality
Entanglement as a resource

What is entanglement good for? **Four examples** where entanglement enables otherwise impossible capabilities:

1) **Superdense coding**: communicate 2 bits by sending 1 qubit
   
   Holevo bound shows that impossible w/o entanglement

2) **Teleportation**: communicate 1 qubit by sending 2 bits

3) **Violating Bell inequalities**: produce non-classical correlations

4) **Quantum cryptography**: distill a shared secret key

It is also **necessary** for any quantum computational speedup.
Superdense coding

If Alice and Bob share EPR pair, they can use it to communicate 2 bits by sending 1 qubit!

\[ |\Phi_{AB}^{(00)}\rangle = (|00\rangle + |11\rangle) / \sqrt{2} = (I \otimes I)|\Phi^+_AB\rangle \]
\[ |\Phi_{AB}^{(01)}\rangle = (|00\rangle - |11\rangle) / \sqrt{2} = (Z \otimes I)|\Phi^+_AB\rangle \]
\[ |\Phi_{AB}^{(10)}\rangle = (|10\rangle + |01\rangle) / \sqrt{2} = (X \otimes I)|\Phi^+_AB\rangle \]
\[ |\Phi_{AB}^{(11)}\rangle = (|10\rangle - |01\rangle) / \sqrt{2} = (XZ \otimes I)|\Phi^+_AB\rangle \]

“Bell basis”

4 orthogonal states created by local operation

“beating” the Holevo bound!
Teleportation

If Alice and Bob share EPR pair, they can use it to communicate 1 qubit by sending 2 bits!

Why does it work? If outcome \( x=\text{z}=0 \), post-measurement state:

\[
\begin{align*}
\Psi &\rightarrow M \\
\psi &\rightarrow \Phi_{\text{MA}}^+ \\
\Phi_{\text{AB}}^+ &\rightarrow ??? \\
\Phi_{\text{MA}}^+ &\rightarrow M \\
\end{align*}
\]

\[
= \left( \langle \Phi_{\text{MA}}^+ | \otimes I_B \right) \left( |\Psi_M\rangle \otimes |\Phi_{\text{AB}}^+\rangle \right) \\
= \frac{1}{2} \sum_{j,k} \left( \langle j|_M \otimes \langle j|_A \otimes I_B \right) \left( |\Psi_M\rangle \otimes |k\rangle_A \otimes |k\rangle_B \right) \\
= \frac{1}{2} I_{M \rightarrow B} |\Psi_M\rangle = \frac{1}{2} |\Psi_B\rangle
\]
Nonlocal correlations

Clauser–Horne–Shimony–Holt

Alice and Bob play CHSH game:

Alice and Bob play CHSH game:

Local classical strategy: \( a = a(x), \ b = b(y) \)

\[
\begin{align*}
\text{Referee} & \\
\text{Alice} & \xrightarrow{x} \quad \text{Referee} \\
& \quad \quad \text{Alice} \\
& \quad \quad \quad \quad \phantom{a} \\
& \quad \quad \quad \quad \phantom{b} \\
\text{Bob} & \xrightarrow{y} \\
& \quad \quad \text{Referee} \\
& \quad \quad \quad \quad \phantom{a} \\
& \quad \quad \quad \quad \phantom{b} \\
\end{align*}
\]

\[
\begin{align*}
a(0) \oplus b(0) & \oplus a(0) \oplus b(1) \oplus a(1) \oplus b(0) \oplus a(1) \oplus b(1) \equiv 0
\end{align*}
\]

⇒ will get at least one answer wrong: \( p_{\text{win}} \leq \frac{3}{4} \)

This is a Bell inequality – a bound on classical correlations!

shared randomness does not help
Nonlocality and quantum cryptography

If Alice and Bob share an EPR pair, they can do better and achieve

$$p_{\text{win},q} = 0.85.$$ 

Tsirelson: optimal

strategy “unique” (rigidity)

- can certify entanglement from correlations alone!

Application: In quantum key distribution, Alice and Bob want to create a key secret from everyone else.

1) Play nonlocal game to ensure that state $$|\Phi^+_{AB}\rangle$$ by rigidity
2) Then $$|\Psi_{ABE}\rangle = |\Phi^+_{AB}\rangle \otimes |\psi_E\rangle$$ by monogamy
3) Now measure EPR pair to get random secret bit.

Very rough sketch!
3. Entanglement in Mixed States

Literature: Lecture notes “Symmetry and Quantum Information”,
https://staff.fnwi.uva.nl/m.walter/qit18/
Entanglement in mixed states

Recall that a state is **separable** if mixture of product states:

\[ \rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \]

not canonical, typically non-orthogonal

**Bad news:** NP-hard to check if \( \rho_{AB} \) separable

\[ \Rightarrow \text{ no entanglement measure is faithful and easy to compute} \]

A practical problem – meaningful calculations are difficult.

Similarly, **multipartite entanglement.** \( \rho_{AB} \) vs purification \( |\Psi_{ABC}\rangle \)
Bound entanglement

Can create any entangled state by LOCC given enough Bell pairs.

Bad news: Transformation usually \textit{irreversible}.

There even exist \textit{“bound entangled”} states such that no Bell pairs can be obtained from any number of copies!

Zoo of entanglement measures: entanglement cost $E_C$, distillable entanglement $E_D$, ...
**PPT criterion**

Idea: Necessary for separability $\iff$ sufficient for entanglement

**Partial transpose (PT):**

$$\langle ab | \rho_{AB}^\Gamma | a'b' \rangle = \langle ab' | \rho_{AB} | a'b \rangle$$

"partial time reverse"

If $\rho_{AB}$ separable then $\rho_{AB}^\Gamma$ is again a density operator.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \implies \rho_{AB}^\Gamma = \sum_i p_i \rho_A^{(i)} \otimes (\rho_B^{(i)})^T$$

**PPT criterion:**

$\rho_{AB}^\Gamma$ negative eigenvalues $\Rightarrow$ $\rho_{AB}$ entangled

E.g. $\Phi^+ \otimes \Phi^+ 1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\Gamma} \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$
Negativity

Partial transpose has \(\text{tr}=1\). Thus, has negative eigenvalues \(\Leftrightarrow\) sum of absolute eigenvalues is \(>1\).

**Negativity:**
\[
N(\rho) = (\Sigma_i |\lambda_i| - 1)/2
\]

**Logarithmic negativity:**
\[
E_N(\rho) = \log \Sigma_i |\lambda_i|
\]

How to calculate?

1) Compute “Renyi negativities” \(\text{tr} (\rho_{AB}^\Gamma)^{2n}\) and let \(n \to 1/2\)

2) Use replica trick: \(\text{tr} (\rho_{AB}^\Gamma)^{2n} = \text{tr} (\rho_{AB}^\Gamma)^{\otimes 2n} (C_{2n} \otimes C_{2n}^{-1})\)

**Feasible** in field theory and holography!

Extendibility criterion

Say $\rho_{AB}$ has \textit{k-extension} if there is state $\sigma$ on $AB_1\ldots B_k$ with

$$\rho_{AB} = \sigma_{AB_1} = \cdots = \sigma_{AB_k}$$

If $\rho_{AB}$ separable then has k-extension for all $k$.

$$\rho_{AB} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \quad \Rightarrow \quad \sigma_{AB_1\ldots B_k} = \sum_i p_i \rho_A^{(i)} \otimes \rho_B^{(i)} \otimes \cdots \otimes \rho_B^{(i)}$$

Conversely, if k-extension then $O(1/k)$ to separable.

\begin{center}
\framebox{Criterion:} $\rho_{AB}$ separable $\iff$ has k-extension for all $k$
\end{center}

$\Rightarrow$ Entanglement is \textit{monogamous} also for mixed state!
**Bonus: De Finetti theorem**

Suppose that \( A_1 \ldots A_n \) is permutation-symmetric. Then reduced states are close to mixtures of product states:

\[
\rho_{A_1 \ldots A_k} \approx \int d\sigma \ p(\sigma) \ \sigma^{\otimes k} \quad \text{if } k \ll n
\]

e.g. \(|00\ldots0\rangle + |11\ldots1\rangle\) and any \(k < n\)

→ another version of monogamy

→ justifies for why in mean field theory it suffices to consider product states
Bonus: Squashed entanglement

While mutual information is not a good entanglement measure, we can construct one using the conditional mutual information:

\[ I(A:B|C) = I(A:BC) - I(A:C) = S(AC) + S(BC) - S(ABC) - S(C) \geq 0 \]

**Squashed entanglement:**

\[ E_{sq}(A:B) = \frac{1}{2} \min_{\rho_{ABC}} I(A:B|C) \]

**Intuition:** entanglement = correlations that cannot be shared

**Properties:**

1. \( 0 \leq E_{sq} \leq \frac{1}{2} I(A:B) \leq \log \min(d_A, d_B) \)
2. For pure states: \( E_{sq} = \frac{1}{2} I(A:B) = S_E \)
3. **Separable** \( \iff \ E_{sq} = 0 \)
4. **Monogamy:** \( E_{sq}(A:B) + E_{sq}(A:C) \leq E_{sq}(A:BC) \)

**Homework:**
Show all but \( \Leftarrow \) in 3.
4. Entanglement in Field Theory

Do quantum information tools apply to quantum field theory?

Challenge: Basic notions such as subsystems, entanglement, entropy, ... more subtle!

Theoretical insights: $c$-theorem from strong subadditivity, Bekenstein bound from relative entropy, renormalization vs QEC...

Another motivation: Quantum computers can simulate quantum mechanics. Can we simulate QFTs or even quantum gravity...?
Subsystems in relativistic QFT

Causal domain of $A$:

$D(A) = \{ p : \text{every maximal causal curve through } p \text{ intersects } A \}$

$\Sigma$ is Cauchy slice if acausal and $D(\Sigma) = \text{everything}$.

Time slice axiom:

$\Sigma \iff \text{global state} \iff \text{Hilbert space } H$

$A \subseteq \Sigma \iff \text{reduced state in } D(A) \iff \text{"H = } H_A \otimes H_B \text{"}$

$D(A) = D(A') \Rightarrow \rho_A \text{ and } \rho_{A'} \text{ should be unitarily related}$
Correlations in QFT

Consider e.g. free scalar field with mass $m$ in Minkowski space:

$$H = \int d^3x \, \pi(x)^2 + (\nabla \phi(x))^2 + m^2 \phi(x)^2$$

$$[\pi(x), \phi(y)] = i\delta^3(x-y)$$

Correlation functions:

$$\langle \phi(x) \rangle = 0$$

$$\langle \phi(x)\phi(y) \rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \exp(-|x-y|/\xi) & \text{if } |x-y| \gg \xi \end{cases}$$

Amusing to compare with Bell pair:

$$\langle X \rangle = \ldots = \langle Z \rangle = 0$$

$$\langle XX \rangle = \ldots = \langle ZZ \rangle = 1$$

General form (short-distance power law, long-distance decay) believed to hold in any relativistic QFT. If $m=0$, decay can be power law.
Entanglement in QFT

Correlation functions:

\[ \langle \phi(x)\phi(y) \rangle \propto \begin{cases} |x-y|^{-2} & \text{if } |x-y| \ll \xi \\ \approx 0 & \text{if } |x-y| \gg \xi \end{cases} \]

Thus, might expect that entanglement entropy satisfies an area law:

\[ S(A) \propto \frac{|\partial A|}{\varepsilon^{d-2}} \]

More generally, might expect that all divergences arise from local integrals over entangling surface \( \partial A \).

That is, assuming \( \xi < \infty \). E.g. for CFTs in \( d=1+1 \), power law decay leads to \( \log(|A|/\varepsilon) \) divergence, as we will discuss momentarily.
Entanglement in QFT

Observables in A, B commute, but Hilbert space does not factorize.

cf. divergence across entangling surface

→ Reduced states not described by density operators
→ Entanglement entropies not obviously well-defined

What can be said rigorously?

→ algebraic QFT literature, Witten’s review

Reeh–Schlieder:

Confusing? No, \( O_A \) will not be unitary!

“\( \{ O_A |\Omega_{AB}\rangle\} \) dense”

Homework: Show that in finite dim any \( |\Psi_{AB}\rangle \) can be written as \( O_A |\Phi_{AB}^+\rangle \).

Relative entropies & various entanglement measures can be rigorously defined and computed/bounded

e.g., still makes sense to distill EPR pairs!

Bisognano–Wichmann: “modular Hamiltonian” of Rindler wedge
We will proceed cavalierly since we must anyways regulate entanglement entropy to obtain finite answer.

General strategy: UV regulate and compute universal quantities

Coefficient of $\log(|A|/\varepsilon)$

Relative entropy

$$D(\rho\|\sigma) = \text{tr} \rho \left( \log \rho - \log \sigma \right)$$

Mutual information $I(A:B)$

If $A$, $B$ don’t touch: “$H_{AB} = H_A \otimes H_B$”

$\Rightarrow$ rigorously defined in QFT!

Intuition: divergences cancel
Euclidean path integrals

Let us consider states that are prepared by Euclidean path integrals. E.g., unnormalized thermal state:

\[ \rho = e^{-\beta H} \]

For \( \beta \to \infty \), obtain vacuum state.

\( \Rightarrow \) Reduced state of \( A \subseteq \Sigma \):

\[ \rho_A = \text{tr}_B e^{-\beta H} \]
Rindler decomposition

Rindler wedges correspond to $A = [0, \infty)$ and $B = (-\infty, 0]$.  

Lorentz boost generator $K$ acts by rotations in Euclidean signature 

$$\rho_A = e^{-2\pi K}$$ 

"thermal"

Similarly, Schmidt decomposition:

$$|\Omega_{AB}\rangle = \Sigma_i e^{-\pi \omega_i} |i'\rangle|i\rangle$$ 

Homework!

Amusing: If $|\Omega_{AB}\rangle$ were product $\Rightarrow$ "firewall" between $A:B$.  

48/114
Entanglement entropy and replica trick

Using the replica trick, it is easy to compute Renyi entropies:

\[
S_n(\rho) = \frac{1}{1-n} \log \frac{\text{tr}[\rho^n]}{\text{tr}[\rho]^n} = \frac{1}{1-n} \left( \log Z_n - n \log Z_1 \right)
\]

where \( Z_n = \text{tr}[\rho^n] = \text{tr}[\rho \otimes_n C_n] \) is calculated by the following path integral:

[Fliss]
Entanglement entropy for single interval

Can be explicitly computed for spherical regions in conformal field theory.

**Cardy-Calabrese:** In 1+1d CFT with central charge $c$, 
\[
S_n = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log \frac{L}{\varepsilon} \quad \quad \quad S = \frac{c}{3} \log \frac{L}{\varepsilon}
\]

Homework: Prove this.

$M_n$ is topologically sphere, compute $Z_n$ from Weyl anomaly.

Alternatively, via 2-point function of twist operators in orbifold CFT:
\[
Z_n = \left\langle \sigma_+ (z_1) \sigma_- (z_2) \right\rangle_{\text{CFT}^n/Z_n}
\]
Application: c-theorem

Can use entanglement entropy to construct RG monotone and re-prove c-theorem.

Suppose we deform “UV CFT” by relevant operator. Then:

\[ S(L \ll \xi) = \frac{c_{UV}}{3} \log \frac{L}{\varepsilon} \]
\[ S(L \gg \xi) = \frac{c_{IR}}{3} \log \frac{L}{\varepsilon'} \]

Claim: \( c(L) = 3 \frac{L}{dS/dL} \) interpolates \( c_{UV}, c_{IR} \) and decreases with \( L \).

Key idea: Use strong subadditivity \( S(AB) + S(BC) \geq S(ABC) + S(B) \).

Here:

\[ S(x) + S(y) \geq S(L') + S(L) \]
\[ = 2S(\sqrt{LL'}) \]

Choose \( L' = L + \delta \) \( \Rightarrow \) \( d^2 \ldots /d\delta^2 \propto -d\text{c}/dL \geq 0 \)
5. Entanglement in Holography

Black holes have a thermodynamic temperature and entropy. This entropy is proportional to the area of the event horizon:

\[ S_{BH} = \frac{\text{Area}}{4G} \]

Bekenstein-Hawking

Surprising! Further puzzles arise when we try to quantize: **Hawking radiation**, **information paradox(es)**, ...

A theory of **quantum gravity** ought to give microscopic explanations.
Holographic principle and practice

**Holographic principle**: Can all information in a region of space be represented as “hologram” living on boundary?

**AdS/CFT duality**: Realization in Anti-de Sitter space

**boundary**: $d$-dim conformal field theory (CFT)

**bulk**: $(d+1)$-dim (string) gravity theory

Controlled setup to study quantum gravity; including black holes, wormholes, ...
AdS/CFT Dictionary

Symmetries ✓

Partition functions:

\[ Z_{\text{CFT}} = Z_{\text{string}} \]

“Extrapolate dictionary”:

\[ O(X) = \lim_{r \to \infty} r^\Delta \phi(r, X) \]

can compute CFT correlation functions:

\[ \int D\phi e^{iS_{\text{eff}}} O_1 \cdots O_n = \langle O_1 \cdots O_n \rangle_{\text{CFT}} \]

What is the bulk dual of entanglement entropy?
Ryu-Takayanagi formula

Ryu-Takayanagi (RT): For static space-times, boundary entropies are computed by area of bulk minimal surface homologous to A:

\[ S(A) = \min \frac{|\gamma_A|}{4G} + \ldots \]

Entanglement \( \Leftrightarrow \) Geometry
Example: AdS$_3$

CFT vacuum state $|\Omega\rangle$ is dual to AdS$_3$ bulk:

Pure state: $S(\Sigma) = 0, \ S(A) = S(A^c)$ \checkmark

For an interval of length $L$, recover Cardy formula:

Poincaré coordinates

$$ds^2 = \ell^2/z^2 \ (dx^2 + dz^2 - dt^2)$$

$$|\gamma_A| = 2\ell \log(L/\varepsilon)$$

$$\Rightarrow \quad S(L) = c/3 \log(L/\varepsilon)$$ \checkmark

Homework: Verify this.

minimal geodesics = coordinate semicircles
Example: Multiple subsystems

Two boundary subsystems:

\[
S(AB) = S(A) + S(B)
\]

\[
I(A:B) = 0
\]

“uncorrelated phase”

\[
S(AB) < S(A) + S(B)
\]

\[
I(A:B) > 0
\]

“correlated phase”
Example: BTZ black hole

BTZ\textsubscript{3} black hole solution is dual to CFT\textsubscript{2} thermal state $\rho_\beta$:

$T = 2\pi r_+$

Mixed state: $S(\rho_\beta) = \frac{\text{horizon area}}{4G_N} > 0 \checkmark$

Phase transition in entanglement entropy:

Entanglement shadow: minimal geodesics don’t reach all the way to $r_+$. 

$S(\rho_\beta)$
Example: Thermofield double

\[ |\text{TFD}_\beta\rangle = \frac{1}{Z} \sum_E e^{-\beta E/2} |E\rangle \langle E'| \]

Thermofield double state is purification of thermal state to two CFTs. Bulk dual: **Two-sided black hole** in static asymptotic AdS space-time.

cf. Rindler wedge analysis
Why should Ryu-Takayanagi hold?

Intuitive generalization of Bekenstein-Hawking formula.

Matches CFT calculations. ✓

Proved under plausible assumptions. ✓

Satisfies many nontrivial consistency checks. For example, easy to verify strong subadditivity:

\[ S(AB) + S(BC) = S(A) + S(B) \geq S(B) + S(ABC) = S(B) + S(ABC) \]

However, we can prove “too much”...
Holographic entropy laws

Ryu-Takayanagi formula satisfies non-standard entropy inequalities. These are constraints for CFTs to have a gravity dual!

"Monogamy" inequality: \( I(A:B) + I(A:C) \leq I(A:BC) \)

Does not hold general states – not even for all probability distributions. Correlations are not monogamous!

\[ \sum_n e^{-\beta E_n/2} |n\rangle \langle n| |n\rangle \langle n| \neq \]

\( \rightarrow \) can be used to witness multipartite correlations

"=" for multipartite correlated states \( \rho_{AB} \otimes \rho_{AC} \otimes \rho_{BC} \)
How to prove holographic entropy inequalities?

\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]

General method that abstracts inclusion/exclusion reasoning:

“Homology regions” for LHS minimal surfaces partition bulk into \( 2^{\text{LHS}} \) regions. ⇒ Hypercube:

- vertices = bulk regions
- edges = surfaces between regions

⇒ use subsets of hypercube to define homology regions for RHS surfaces
  
  not necessarily minimal

Homework:

Work out details.

If each edge cut at most once: Entropy inequality is valid!
To illustrate the method, let us prove the “monogamy inequality”, which expands to:

\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

Infinitely many holographic entropy inequalities can so be proved. How to organize systematically?
Holographic entropy cones

For fixed number of subsystems, consider all possible entropy vectors:

\[ C_n = \{ S_{RT}(A_1), \ldots, S_{RT}(A_1A_2\ldots A_n) \} \]

arbitrary geometries allowed!

This is a polyhedral convex cone – the holographic entropy cone.

faces: entropy inequalities such as \( S(A) + S(B) \geq S(AB) \)

rays: entropy vectors that cannot be written as mixture of others. represented by “extremal geometries”.

can these be identified with microscopic building blocks??
Constraints from entropy inequalities

Can also go the other way and exploit known entropy inequalities to derive gravitational constraints. E.g., using relative entropy:

$$S(\rho \parallel \sigma) = \text{tr} \rho \log \rho - \text{tr} \rho \log \sigma \geq 0$$

Perturb around vacuum state:

- 1st order: linearized Einstein equations
  Faulkner et al
- 2nd order: positive energy inequalities
  Lin et al, Lashkari et al

  e.g. $$\int T_{00} \sqrt{g} \geq 0$$

Much more to be said about holographic entropies (monotonicity of relative entropy, Freedman-Headrick bit threads, ...)
Generalizations

Entropy of bulk fields in region enclosed by RT surface contribute $O(1)$ corrections to entropy:

$$S(A) = \frac{|\gamma_A|}{4G} + S(\alpha)$$

better: minimize joint expression ("generalized entropy")

$$S(A) = \max_{\Sigma} \min_{\gamma_A} \frac{|\gamma_A|}{4G}$$

RT holds in **static situations** (more generally, in time-reflection symmetric situations). In general, consider **extremal area codimension-2 spacelike bulk surfaces**.

Hubeny-Rangamani-Takayanagi (HRT)

Equivalently, Wall’s maximin procedure:

$$S(A) = \max_{\Sigma} \min_{\gamma_A} \frac{|\gamma_A|}{4G}$$

Faulkner-Lewkowycz-Maldacena

Engelhardt-Wall, ...
6. Toy Models of Holography

Holography is mysterious...

1) “Extrapolate” dictionary: \( r^A \phi(X,r) \xrightarrow{r \to \infty} O(X) \)

A puzzle: \([\phi(y), O(X)] = 0\) !?

2) Ryu–Takayanagi with bulk corrections:

\[
S(A) = \min \left| \frac{|\gamma_A|}{4G} + S(a) \right|
\]

3) Bulk reconstruction problem: Every bulk operator should be dual to some boundary operator.

\[
\phi(x) \overset{!?}{=} \int O(X) K(X|X(x)) dX
\]

Why do we care? Extrapolate dictionary insufficient if want to study processes behind horizons, understand bulk locality.
Subregion duality:

Can write any bulk operator in \( a \) as boundary operator in \( A \)!

Proved using QI tools. ✓ Dong-Harlow-Wall, Cotler-…-W

Not known how to do explicitly in most tantalizing situations:

Only when \( A = \) everything or \( \phi(x) \) in (smaller) causal wedge of \( A \).

⇒ Hamilton-Kabat-Lifschytz-Lowe, Banks et al, Heemskerk et al, …, Harlow TASI
Holography is mysterious

Subregion duality leads to another puzzle:

\[ \phi = O_{AB} = O_{AC} = O_{BC} \]

no common support \( \nsubseteq \)
\[ AB \cap AC \cap BC = \emptyset \]

Resolution: Only "few" states correspond to any particular semiclassical bulk description.

"[\phi(y), O(X)] = 0" or "O = \phi" only hold (make sense!) on small subspaces of CFT Hilbert space, known as "code subspaces"

Plan: Discuss toy models that reproduce 1)-3) and resolve puzzles by simple QI mechanisms.
Three-Qutrit code

\[ C^3 \rightarrow C^3 \otimes C^3 \otimes C^3 \]

\[ V|i\rangle = |\tilde{i}\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j+i, j-i\rangle \]

encodes 3-dim in 27-dim space

states \( \rho \) are encoded by \( \tilde{\rho}_{ABC} = V\rho V^\dagger \)

operators \( \phi \) are encoded by \( \tilde{\phi}_{ABC} = V\phi V^\dagger \)

Key fact: \[ V|i\rangle = (I_A \otimes U_{BC})(|\Phi^+_{AB}\rangle \otimes |i_C\rangle) \]

where \( U_{BC}|j,i\rangle = |j+i, j-i\rangle \)

\[ \langle \Phi\ldots \rangle_\rho = \langle \tilde{\Phi}\ldots \rangle_{\tilde{\rho}} \]
Three-Qutrit code

This has remarkable consequences:

Ryu-Takayanagi:

\[ S(A) = \log(3) \]
\[ S(AB) = \log(3) + S(\rho) \]
\[ = S(B) = S(C) \]
\[ = S(AC) = S(BC) \]

Subregion duality: can decode \( \rho \) from BC alone!

\[ \text{Heisenberg picture: } O_{BC} = U_{BC}(I \otimes \Phi)U_{BC}^* \]
\[ \Rightarrow O_{BC}V = V\Phi, \quad O_{BC}^*V = V\Phi \]

\[ \Rightarrow \text{"erasure code"}: \text{can correct for loss of single qutrit!} \]

\[ \Rightarrow \text{resolves second puzzle!} \]

\[ \Rightarrow O_{BC} = \Phi \]
Three-Qutrit code

Similarly, if $\phi$ is any bulk and $O_A$ any boundary operator on $A$:

$$\langle i | [O_A, \Phi_{ABC}] | j \rangle = \langle i | [O_A, O_{BC}] | j \rangle = 0$$

$\Rightarrow$ resolves first puzzle!

Quantum error correction plays important role in recent research in holography (emergence of bulk locality, black hole information paradox, ...)

Verlinde$^2$, Almheiri-Dong-Harlow, ...
7. Error Correction, Decoupling, and Black Holes
Recall: Quantum channels

**Quantum channel:** Any combination of unitary evolution, partial traces, adding auxiliary systems.

Equivalently, any map that sends states $\rho_{AR} \rightarrow$ states $\rho_{BR}$.

$$\rho_{BR} = (T \otimes \text{id})(\rho_{AR})$$

completely positive & trace-preserving (CPTP)

Examples:

- **Basis measurement:**
  $$M(\rho) = \sum_x \langle x|\rho|x\rangle |x><x|$$

- **Depolarizing noise:**
  $$D_p(\rho) = p\rho + (1-p)I/d$$

Homework: Check this.
Tools for quantum channels

**Choi state:** characterizes channel completely!

\[
\Omega_{AB} = (\text{id} \otimes T)(\Phi_{A'A}^+)
\]

**Stinespring extension:** Isometry \( V \) such that:

\[
T(\rho) = \text{tr}_E(V \rho V^+) \]

⇒ complementary channel:

\[
T_c(\rho) = \text{tr}_B(V \rho V^+) \]

Together: Solve channel problems by (pure) state reasoning!
Example: Basis Measurement

\[
M(\rho) = \sum_x \langle x|\rho|x\rangle \ |x><x|
\]

\[
\Omega_{A'B} = (\text{id} \otimes M)(|\Phi^+_{A'A}\rangle\langle\Phi^+_{A'A}|) = 1/d \sum_{x,y} (\text{id} \otimes M)(|xx><yy|)
\]

\[
= 1/d \sum_{x,y} |x><y| \otimes M(|x><y|) = 1/d \sum_x |x><x| \otimes |x><x|
\]

\[
= 1/d \sum_x |xx><xx|
\]

\[
V|x\rangle = |xx\rangle
\]

\[
\text{tr}_E(V|x><y|V^\dagger) = \text{tr}_E(|xx><yy|) = \delta_{xy} |x><x| = M(|x><y|)
\]

Complementary channel: \( M^c = M \)

Homework: Compute Choi + Stinespring for other examples.
Quantum error correction

When building quantum computers, we want to protect against errors (imperfections, noise, decoherence, ...).

To achieve this, redundantly encode "logical" into "physical" qubits:

For example, 3-qutrit code corrects against erasure of any 1 qutrit.

Holography: bulk d.o.f \rightarrow \text{dictionary} \rightarrow \text{boundary} \rightarrow \text{trace over part of bdry}

Questions:

1) When can we in principle correct?
2) How to correct in practice?
Decoupling criterion

The question: Given a channel $T_{A\rightarrow B}$, when can we reverse it?

**Decoupling criterion:** Can reverse $T_{A\rightarrow B}$ if and only if the complementary channel $T^c_{A\rightarrow E}$ is constant.

- exactly what we found for 3-qutrit code
- very strong form of “no cloning” statement

If reversible: There exists state $|\chi\rangle$ and isometry $W$ such that:

$$V_{AF}W_{AB} = \Omega_{A'E} = \Omega_{A'} \otimes \chi_E$$

$$I(A':E) = 0$$

or $|\Omega_{A'AEF}\rangle = |\Phi^+_{AA'}\rangle \otimes |\chi_{EF}\rangle$

Homework: Prove this.
Teleportation revisited

It is instructive to revisit teleportation from this perspective.
Consider channel which performs Bell measurement on $\rho_M \otimes I_A/2$:

This is a constant channel since all outcomes are equally likely. By the decoupling criterion, can decode from complementary channel!

First, compute Stinespring extension:

$$U |\Phi^{(xz)}\rangle = |xzxz\rangle$$

Thus, complementary channel looks like teleportation w/o correction:
Decoupling inequality

In information theory, random codes are often almost optimal.

When can we decode $A$ from $B$?

The following result addresses these kind of problems:

**Decoupling Inequality:** Let $\rho_{ABE}$ state, $U_{BE}$ random. Then:

$$
\int dU_{BE} \left\| \text{tr}_B (U_{BE} \rho_{ABE} U_{BE}^+) - \rho_A \otimes I_E / d_E \right\|_1^2 \leq \frac{d_{AE}}{d_B} 2^{-S_2(\rho)}
$$
Hayden-Preskill protocol

We again model an evaporating black hole by random unitary. After Page time, assume black hole maximally entangled with old radiation.

Now suppose Alice throws her diary into black hole.

How much further radiation do we need to collect so that we can recover diary?

That is, when can we decode A from RR′?

Need $\Omega_{A'B} \approx \Omega_{A'} \otimes \Omega_B$!

**Answer:** $d_A \ll d_R$

Little more than size of diary – independent of size of black hole. Black hole after Page time is like a mirror, information comes right out.

Homework: Show this using the decoupling theorem with $B \rightarrow R, E \rightarrow B$. 

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Holographic teleportation

Wormhole in thermofield double state can be made traversable by weak local classical coupling between the two CFTs.

This holographic teleportation protocol is remarkable: “self-decoding” even though CFT time evolution scrambling!

Recent work constructed toy models using e.g. random unitaries and proposed general QI mechanisms.

Effective interaction only depends on operator sizes.
**Bonus: Relative entropy**

\[
D(\rho \| \sigma) = \text{tr} \rho (\log \rho - \log \sigma) \geq 0 \quad \text{iff } \rho = \sigma
\]

well-defined in QFT

\[ S(\rho) = \log d - D(\rho \| I/d), \quad I(A:B) = D(\rho_{AB} \| \rho_A \otimes \rho_B), \ldots \]

Pinkser inequality: \[ D(\rho \| \sigma) \geq \frac{1}{2 \ln 2} \| \rho - \sigma \|_1^2 \]

Data processing inequality: \[ D(\rho \| \sigma) \geq D(T(\rho) \| T(\sigma)) \Rightarrow \text{strong subadditivity} \]

"=" \[ \Leftrightarrow \text{can reverse channel on pair of states } \rho, \sigma \]

How so? Use Petz map: \[ D(\rho) = \sigma^{1/2} T^\dagger (T(\sigma)^{-1/2} \rho T(\sigma)^{-1/2}) \sigma^{1/2} \]
Back to our toy model

Toy model has no geometry – just a single site!

To go beyond, let’s focus on the RT formula:

\[ V|i\rangle = |\tilde{i}\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j + i, j - i\rangle \]

\[ |\tilde{0}\rangle = \frac{1}{\sqrt{3}} \sum_{j=0}^{2} |j, j, j\rangle \]

Can we glue together many such states (or codes)?
8. Tensor Network Toy Models

Many-body quantum states have exponentially large description

\[ |\psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

tensor with n indices

In practice: entanglement is local, correlations decay rapidly

→ can hope for more efficient description:

Key idea: start with entangled pairs...

...and apply local transformations:

\[ |0\ldots00\rangle + |1\ldots11\rangle \text{ from } |00\rangle \rightarrow |0\rangle, \ |11\rangle \rightarrow |1\rangle \]

e.g. ‘cat’ state |0\ldots00\rangle + |1\ldots11\rangle from |00\rangle \rightarrow |0\rangle, \ |11\rangle \rightarrow |1\rangle
Tensor networks as a tool

**Tensor network**: state defined by contracting network of (local) tensors

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_n} \psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle \]

- **MPS**: \( \ldots \)
  - White, Fannes-Nachtergaele-Werner, Östlund-Rommer

- **PEPS**: \( \ldots \)
  - Verstraete-Cirac

**Numerical tool**: efficient variational classes

- Provably so for gapped theories in 1+1d (Hastings)
- Can even have interpretation as quantum circuits

**Powerful theoretical formalism**, provides “dual” descriptions of complex phenomena → quantum phases, topological order, ...
Computing with tensor networks

Very similar to path integral reasoning:

\[
\langle \psi | \psi \rangle = \quad \begin{array}{c}
\text{tensor network 1} \\
\text{tensor network 2}
\end{array}
\quad = \quad \begin{array}{c}
\text{graph 1} \\
\text{graph 2}
\end{array}
\]

\[
\langle \psi | \Theta | \Psi \rangle = \quad \begin{array}{c}
\text{tensor network 1} \\
\text{tensor network 2}
\end{array}
\quad = \quad \begin{array}{c}
\text{graph 1} \\
\text{graph 2}
\end{array}
\]

Can formally obtain tensor networks by trotterizing \( e^{-\beta H} \).

What is the role of the network geometry?
Entanglement entropy satisfies “Ryu-Takayanagi bound”:

\[ S(A) \leq N |\gamma_A| \]

N qubits/bond
\[ \gamma_A = \text{minimal cut} \]

In general, the bound is not saturated...

Tantalizing: Picture shows Vidal’s MERA tensor network.
Used for critical theories, it looks like a time slice of AdS!
Why does the bound hold?

\[ |\Psi_{AB}\rangle = \]

Thus, the Schmidt rank is at most \(2^N |\gamma_A|\).

\[ S(A) \leq S_0(A) \leq N |\gamma_A| \]

NB: Bound saturated if L, R are unitaries (or isometries)!
Holography from tensor networks

Want “exactly solvable” toy models of holographic duality:

Approach: Define boundary state via tensor network in bulk

- simple bulk tensors, e.g. random and large $N$
- emergent Ryu-Takayanagi law!

$S(A) \approx N |\gamma_A|$
Assume each local tensor is **perfect** = isometry in all possible directions.

Choose **orientations** such that $\gamma_A \to A, A^c$

Then: $V, W$ isometries and **RT formula holds**

Always possible for graphs with “negative curvature” and A “single interval”.

Concrete and intuitive!

How to generalize?
Random tensor model

Choose random bulk tensors of large bond dimension.

⇒ emergent RT formula

Three interpretations:

1. Random tensors ≈ perfect
2. Entanglement distillation protocol
3. Disorder average ⇒ ferromagnetic spin model

large $N$ ⇒ low $T$
Derivation of Ryu–Takayanagi law

Setup:

### Setup:

Arbitrary lattice or graph. Tensors are chosen i.i.d. from Haar measure.

Recall: In any tensor network: \( S(A) \leq N |\gamma_A| \).

#### Strategy:

**Lower bound** \( S_2(A) \) using replica trick.

\[
|\Psi\rangle = \left( \bigotimes \langle xy| \right) \left( \bigotimes_x |V_x\rangle \right)
\]

max. entangled states

random tensors

|xy\rangle = \sum_{\mu=1}^{D} |\mu, \mu\rangle
Replica trick for 2\textsuperscript{nd} Rényi

\[ S_2(A) = -\log \text{tr}[\rho_A^2] \]

\[ \langle \Psi | = \left( \bigotimes_x \langle x y| \right) \left( \bigotimes_{x,y} | V_x \rangle \right) \]

Pick \( I \) vs \( F \) at each vertex.

Each loop is trace: factor \( D = 2^N \)

Ising variables & boundary conditions!
2nd Rényi entropy

\[ S_2(A) = \log \text{tr}[\rho_A^n] \]

\[ \text{tr}[\rho_A^2] \approx \text{partition function of ferromagnetic Ising model at } 1/T = \log(D) \]

\[ S_2(A) \approx -\log \text{tr}[\rho_A^2] \approx \log(D) |\gamma_A| \]

Ryu-Takayanagi formula!
What does it mean?

Random tensor networks (RTN) provide intuitive toy model. Reproduce Ryu-Takayanagi formula (+ much more). Analyzed using replica trick.

Relevance for holography? Ryu-Takayanagi formula is proved similarly. But: Einstein equations $\Rightarrow$ nontrivial spectrum!

Is all hope lost? No! Remarkably, RTN match precisely so-called fixed-area states in holography.

Moreover, general states can be expanded in terms of fixed-area states. Under certain “diagonal approximations”, can lift results!

Similarly, random quantum circuit models have recently been studied, exhibit interesting phenomenology. Relevant to “quantum supremacy” proposals etc.
Bonus: Entanglement of assistance

Multiparty entanglement distillation: create entanglement between Alice & Bob with help of Charlies by measurements & classical communication.

|ψ⟩ = \((\bigotimes_x |V_x⟩)(\bigotimes_{x,y} |xy⟩)\)

measurement in random basis

optimal! merges state w.h.p.

General mechanism for producing Ryu-Takayanagi from area law state!
**Holographic mappings**

AdS/CFT is duality between two theories = “dictionary” that maps states & observables. How to incorporate into toy model?

**Approach:** Define **bulk-boundary** mapping via **tensor network**

= combination of both toy models

**red legs:** bulk degrees  
**black legs:** boundary degrees

“logical” bulk states are **encoded** in “physical” boundary Hilbert space

Toy model of how bulk quantum fields get encoded in boundary CFT
Holographic codes

If bulks legs have small dimension $d \ll D$, obtain error correcting code that satisfies “subregion duality”, a key QI feature of AdS/CFT:

Bulk degrees of freedom in $a$ ($b$) get encoded into $A$ ($B$)!

In particular, bulk corrections to entropy: $S(A) \approx N |\gamma_A| + S(a)$
1) **HaPPY argument**: Choose orientations s.th. \( a\gamma_A \rightarrow A, b\gamma_A \rightarrow B \).

Interpretation: Holographic codes are macroscopic erasure codes built from microscopic ones (perfect tensors).

2) **Decoupling argument**: Only need to prove that \( I(a':b'B) = 0 \). Why? See later!

\( \rightarrow \) Can prove geometrically since Choi state satisfies Ryu-Takayanagi!
Subregion duality from decoupling

By decoupling, suffices to prove that \( I(a:bB) \approx 0 \) in Choi state:

Schematically:

\[
S(a) = \log(d) \, |a| \\
S(bB) = \log(D) \, |\gamma_A| \\
S(abB) = \log(D) \, |\gamma_A| + \log(d) \, |a|
\]

Assume bulk legs have small dimension \( d \ll D \).
Quantum minimal surfaces and islands

What if bulk entropy is not small?

\[ S_2(A) \approx \min \{ N |\gamma_A| + S_2(a) \} \]

“Quantum minimal surface”, minimizes “generalized entropy”.

Proof using replica trick (additional action from bulk state)!

E.g., if we add highly entangled state between distant bulk sites, obtain “island” disconnected from boundary.

Holographic counterparts feature crucially in very recent developments on black hole information paradox that seek to give a bulk picture of black hole evaporation.

Surprising that the simple RTN model reproduces these features!?
9. Subregion Duality and Subsystem Error Correction

Literature: https://arxiv.org/abs/1607.03901
Subregion duality

Let us talk more systematically about the quantum information structure of subregion duality. Consider an isometry:

\[ V_{AB} \]

Notation:

\[ \rho_{ab} \text{ state } \Rightarrow \rho_{AB} = V \rho_{ab} V^\dagger \]

Subsystem error correction: When can we recover \( a \) from \( A \)?

More subtle than what we discussed last lecture. There we had no “b” system – now \( \rho_{ab} \) can be correlated or entangled!
Subsystem error correction

The following conditions are equivalent:

1) There is a channel $D_{A \rightarrow a}$ such that:
   $$D(\rho_A) = \rho_a \text{ for all } \rho_{ab}$$

2) For all $\phi_a$, exists $O_A$ such that:
   $$V\phi_a = O_A V \text{ and } V\phi_a^\dagger = O_A^\dagger$$

3) For all $\phi_a$, and $XB$:
   $$[\phi_a, V^\dagger XB V] = 0$$

4) Decoupling:
   $$I(a':b'B) = 0$$
   i.e. $$\Omega_{A'b'B} = \Omega_{A'} \otimes \Omega_{b'B}$$

Aside: 2) allows computing correlation functions – even if we use different subsystems for each operator:

$$\langle \phi_{ab} \phi'_{bc} \rangle = \langle O_{AB} O'_{BC} \rangle$$
Proof sketch of equivalence

1) There is a channel $D_{A \rightarrow a}$ such that:

$$D(\rho_A) = \rho_a \text{ for all } \rho_{ab}$$

$\Leftrightarrow$ Stinespring extension:

2) $V\phi_a = O_A V$

$V\phi_a^\dagger = O_A^\dagger V$

$O_A = W^\dagger \phi_a W$

$\Leftrightarrow$ Choi state:

3) $[\phi_a, V^\dagger X_B V] = 0$

$\Leftrightarrow$ compare purifications

4) $I(a':b'B) = 0$
Complementary recovery

When can we recover $a$ from $A$ and $b$ from $B$? Result:

Normal form:

Ryu-Takayanagi formula:

The punchline:

Precisely like in toy models!

$\Rightarrow$ “proof” that latter holds in AdS/CFT.
Bonus: Proof that Ryu-Takayanagi implies complementary recovery

Assume: \( S(A) = c + S(a) \) for all \( \rho_{ab} \)

Use 1\textsuperscript{st} law: \( \text{tr}[K_a \delta \rho_{ab}] = \delta S(a) = \delta S(A) = \text{tr}[K_A \delta \rho_{AB}] = \text{tr}[V^* K_A V \delta \rho_{ab}] \)

\[ V^* K_A V = c + K_a \]

for modular Hamiltonians \( K_a = -\log \rho_a \), \( K_A = -\log \rho_A \)

\[ D(\rho_A || \sigma_A) = -\text{tr}[\rho_A K_{\rho A}] - \text{tr}[\rho_A K_{\sigma A}] = -\text{tr}[\rho_a V^* K_{\rho A} V] - \text{tr}[\rho_a V^* K_{\sigma A} V] = \ldots \]

\[ D(\rho_A || \sigma_A) = D(\rho_a || \sigma_a) \] for all \( \rho_{ab} \) and \( \sigma_{ab} \)

\[ \rho_{ab} = e^{i\phi_{bs}} \sigma_{ab} e^{-i\phi_{bs}} \]

\[ [\phi_b, V^* X_A V] = 0 \]

Use Petz map to obtain decoder \( D_{A \rightarrow a} \)

Homework: Fill in the details.
Decoding the hologram (using error correction)

This proof of “entanglement wedge” reconstruction property is **nonconstructive & nonrobust**

How to find **boundary reconstruction of local bulk operator**?

Banks et al, Hamilton et al, Kabat et al, Heemskerk et al, Lin et al, Faulkner–Lewkowycz, ...

Recall: Only understood in special cases.

Recent progress in theory of quantum error correction led to **robust** proof. More explicit formulas and **decoding protocols**?

Cotler–…–W, Kitaev–Yoshida, Hayden–Penington
State dependence

Theorem models situation where minimal surface can be considered fixed for all states in code subspace (no backreaction).

In general, state-dependent! “Quantum” minimal surface obtained by minimizing generalized entropy:

\[ S(A) = \min \left\{ \frac{|\gamma_A|}{4G} + S(a) \right\} \]

realized in random tensor network model! ✓

This form of subregion duality has featured crucially in very recent research on the black hole information paradox that seeks to give a bulk picture of black hole evaporation.

→ Penington, Almheira et al, lectures by Netta?
Summary

Whirlwind tour through some key concepts and tools of quantum information, motivated by applications to QFT and holography:

States, Channels, Entropy
Entanglement of Pure and Mixed States
Entanglement in Field Theory and Holography
Toy Models of Holography
Quantum Error Correction and Decoupling

No time for quantum computing: circuits, algorithms, complexity, ... 😞

Slides: https://staff.fnwi.uva.nl/m.walter/