Quantum Brascamp-Lieb Inequalities

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based on joint work with Mario Berta and David Sutter

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Overview

geometric inequalities $\longleftrightarrow$ entropy inequalities

Brascamp-Lieb inequalities have wide range of applications and satisfy beautiful duality. We study a quantum formulation, motivated by the desire to identify new tools to proving entropy inequalities.

Plan for today:

1. Introduction
2. Quantum BL duality, applications and connections
3. Geometric quantum BL inequalities
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Classical Brascamp-Lieb inequalities

For $B_k : \mathbb{R}^m \to \mathbb{R}^{m_k}$ linear, $q_k > 0$, $C > 0$, an inequality of the form

$$\int_{\mathbb{R}^m} \prod_{k=1}^n |f_k(B_k x)| \, dx \leq C \prod_{k=1}^n \|f_k\|_1^{1/q_k} \quad \forall f_k$$

This generalizes many classical integral inequalities (Hölder, Young, ...)
Many proofs, applications, variations...

- Optimal $C$ can be computed by optimizing over Gaussian $f_k$. [Lieb]
- When is $C$ finite? Fully classified. [Bennett et al]
- How to compute $C$ efficiently? Still partly open! [Garg et al]

Geometric case: $B_k$ projections s.th. $\sum_{k=1}^n q_k B_k^* B_k = I_m$. 
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Duality and entropy

BL inequality is dual to ‘subadditivity’ inequality for differential entropy:

\[
\sum_{k=1}^{n} q_k S(B_k X) \geq S(X) - \log C \quad \forall \text{RV } X \text{ on } \mathbb{R}^m
\]

Apart from information theoretic interest, equivalence also enables new proof techniques (heat flow). [Carlen–Cordero-Erausquin]

The duality can be generalized to arbitrary channels and relative entropies. Framework includes hypercontractivity, strong data processing, etc. [Liu et al]

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Result: Quantum Brascamp-Lieb Duality

Let $\mathcal{E}_k: L(\mathcal{H}) \rightarrow L(\mathcal{H}_k)$ positive & TP, $q_k > 0$, $\sigma$, $\sigma_k > 0$, $C > 0$. Then the following are equivalent:

$$\sum_{k=1}^{n} q_k \, D(\mathcal{E}_k(\rho) \| \sigma_k) \leq D(\rho \| \sigma) + \log C \quad \forall \text{ states } \rho$$

and

$$\text{tr} \, e^{\log \sigma + \sum_{k=1}^{k} \mathcal{E}_k^*(\log \omega_k)} \leq C \prod_{k=1}^{n} \| e^{\log \omega_k + q_k \log \sigma_k} \|_{1/q_k} \quad \forall \omega_k > 0$$

- Proof via Legendre: $D(\rho \| \sigma) = \sup_{\omega > 0} \{ \text{tr} \, \rho \log \omega - \log \text{tr} \, e^{\log \omega + \log \sigma} \}$ [Petz]
- Not clear which side looks more intimidating . . .
- Useful choices: $\sigma_k = \mathcal{E}_k(\sigma)$ or $\sigma_k = I$, $\sigma = I$
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When specializing to $\sigma_k = I, \sigma = I$, recover equivalence between

$$\sum_{k=1}^{n} q_k S(\mathcal{E}_k(\rho)) \geq S(\rho) - \log C \quad \forall \text{ states } \rho$$

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For example, can prove uncertainty relations via trace inequalities, as pioneered by Frank-Lieb:

- Maassen-Uffink: $S(X) + S(Z) \geq S(\rho) + 1$ via Golden-Thompson
- Six-state [Coles et al]: $S(X) + S(Y) + S(Z) \geq S(\rho) + 2$ via Lieb 3-matrix
Without side information

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$$\sum_{k=1}^{n} q_k S(\mathcal{E}_k(\rho)) \geq S(\rho) - \log C \quad \forall \text{ states } \rho$$

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$$\text{tr} \left( e^{\sum_{k=1}^{n} \mathcal{E}_k^*(\log \omega_k)} \right) \leq C \prod_{k=1}^{n} \| \omega_k \|_{1/q_k}^{1/q_k} \quad \forall \omega_k \succ 0$$

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Applications and questions

- Can we prove new **uncertainty relations** involving multiple measurements (and even general quantum channels)? $N$-matrix GT?

- **Strong data-processing** inequalities fall into the framework:
  \[
  D(\mathcal{E}(\rho)\|\mathcal{E}(\sigma)) \leq \eta D(\rho\|\sigma) \quad \forall \rho
  \]

- Tensorization holds classically, but fails quantumly:
  \[(\mathcal{E}, C) \& (\mathcal{E}', C') \not\Rightarrow (\mathcal{E} \otimes \mathcal{E}', C \cdot C')\]
  Examples include **non-additivity** of minimal output entropy. Useful?

- Computational complexity of testing validity of (families of) BL ineqs?

- Relation to works by Carlen-Maas?
Back to geometry...

Recall the classical Brascamp-Lieb inequalities in the geometric case:

\[
\sum_{k=1}^{n} q_k S(P_k X) \geq S(X) \quad \forall \text{ RV } X \text{ on } \mathbb{R}^m
\]

with \( P_k \) projections onto subspaces \( V_k \subseteq \mathbb{R}^m \) s.th. \( \sum_{k=1}^{n} q_k P_k = I_m \).

How can we formulate a quantum version? For any subspace \( V \subseteq \mathbb{R}^m \),

\[
L^2(\mathbb{R}^m) = L^2(V \oplus V^\perp) = L^2(V) \otimes L^2(V^\perp)
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hence can define reduced state \( \rho_V \) for any state \( \rho \) on \( L^2(\mathbb{R}^m) \).

This generalizes the usual partial trace. In general, can interpret as state of subset of modes after subjecting \( \rho \) to network of beamsplitters.
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Theorem

Let $P_k$ projections onto subspaces $V_k \subseteq \mathbb{R}^m$ s.th. $\sum_{k=1}^{n} q_k P_k = I_m$. Then, for all states $\rho$ on $L^2(\mathbb{R}^m)$ with finite first and second moments:

$$\sum_{k=1}^{n} q_k S(\rho_{V_k}) \geq S(\rho)$$

- For coordinate subspaces recover quantum Shearer inequality.  [Carlen-Lieb]
- But already nontrivial for “Mercedes star” configuration in $\mathbb{R}^2$:

  ▶️

- Also holds conditioned on side information.  [Ligthard]
- Can generate more ineqs. via Gaussian unitaries: $\text{Sp}_{2m} \circledast L^2(\mathbb{R}^m) \ldots$
Sketch of proof

\[ \sum_{k=1}^{n} q_k S(\rho_{V_k}) \geq S(\rho) \]

Implement classical proof strategy of Carlen–Cordero-Erausquin using quantum heat flow of König-Smith:

\[ \frac{d}{dt} \rho = - \sum_{j=1}^{m} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]] \]

Asymptotic scaling of entropy: \( S(\rho_{V(t)}) \sim \dim V \log t \)

- Inequality holds at \( t = \infty \) if \( \sum_{k} q_k \dim V_k \geq m \).

Quantum de Bruijn identity: \( \frac{d}{dt} S(\rho) = J(\rho) \), a Fisher information.

- Can prove reverse inequality for Fisher information if \( \sum_{k} q_k P_k \leq I_m \):

\[ \sum_{k=1}^{n} q_k J(\rho_{V_k}) \leq J(\rho) \]
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cf. [De Palma–Trevisan]
Gaussian BL beyond the geometric case

There is a natural action of $\text{Sp}_{2m}$ on $L^2(\mathbb{R}^m)$ by Gaussian unitaries. Any symplectic matrix $B \in \mathbb{R}^{2m' \times 2m}$ determines subsystem of $m'$ modes, so we can define reduced state $\rho_B$ on $L^2(\mathbb{R}^{m'})$ for any state $\rho$ on $L^2(\mathbb{R}^m)$.

This notion generalizes the reduced state $\rho_V$ for subspaces $V \subseteq \mathbb{R}^m$ and leads naturally to the following class of Gaussian quantum BL inequalities:

$$\sum_{k=1}^{n} q_k S(\rho_{B_k}) \geq S(\rho) + c$$

where the $B_k \in \mathbb{R}^{2m_k \times 2m}$ symplectic matrices. When does it hold?

Recent result (De Palma–Trevisan): Assuming $\sum_{k=1}^{n} q_k m_k = m$, inequality holds for all quantum states iff holds for all probability densities!
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- Also holds conditioned on side information.
- Can also include “classical” outputs (≈ quadrature measurements)
- Proof again based on quantum heat flow strategy!
Outlook

<table>
<thead>
<tr>
<th>trace inequalities</th>
<th>BL</th>
<th>entropy inequalities</th>
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<tbody>
<tr>
<td></td>
<td>duality</td>
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Duality between quantum relative entropy inequalities and trace inequalities. **Unifying framework** to tackle information theoretic questions. New family of **geometric** quantum Brascamp–Lieb inequalities.

Many exciting directions:

- Uncertainty relations from $n$-matrix GT?
- Sufficient conditions for tensorization?
- Applications of new trace inequalities?
- Other applications of quantum heat flow?
- ...  

*Thank you for your attention!*