

Quantum Brascamp-Lieb Inequalities

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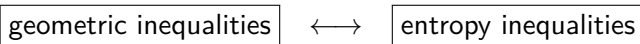


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based on joint work with Mario Berta and David Sutter

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Overview

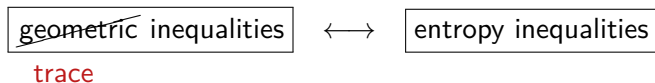


Brascamp-Lieb inequalities have wide range of applications and satisfy **beautiful duality**. We study a **quantum formulation**, motivated by the desire to identify new tools to proving entropy inequalities.

Plan for today:

- 1 Introduction
- 2 Quantum BL **duality**, applications and connections
- 3 **Geometric** quantum BL inequalities

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Classical Brascamp-Lieb inequalities

For $L_k: \mathbb{R}^m \rightarrow \mathbb{R}^{m_k}$ linear, $q_k > 0$, $C > 0$, an inequality of the form

$$\int_{\mathbb{R}^m} \prod_{k=1}^n |f_k(L_k x)| dx \leq C \prod_{k=1}^n \|f_k\|_{1/q_k} \quad \forall f_k$$

This generalizes many **classical integral inequalities** (Hölder, Young, ...)

Many proofs, applications, variations. . .

- ▶ Optimal C can be computed by optimizing over **Gaussian** f_k . [Lieb]
- ▶ When is C finite? Fully classified. [Bennett et al]
- ▶ How to compute C **efficiently**? Still partly open! [Garg et al]

Geometric case: L_k projections s.th. $\sum_{k=1}^n q_k L_k^* L_k = I_m$.

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Duality and entropy

BL inequality is dual to 'subadditivity' inequality for differential entropy:

$$\sum_{k=1}^n q_k S(L_k X) \geq S(X) - \log C \quad \forall \text{ RV } X \text{ on } \mathbb{R}^m$$

Apart from information theoretic interest, equivalence also enables new proof techniques (heat flow). [Carlen–Cordero-Erausquin]

The duality can be generalized to arbitrary channels and relative entropies. Framework includes hypercontractivity, strong data processing, etc. [Liu et al]

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Result: Quantum Brascamp-Lieb Duality

Let $\mathcal{E}_k: L(\mathcal{H}) \rightarrow L(\mathcal{H}_k)$ positive & TP, $q_k > 0$, $\sigma, \sigma_k \succ 0$, $C > 0$.
Then the following are **equivalent**:

$$\sum_{k=1}^n q_k D(\mathcal{E}_k(\rho) \| \sigma_k) \leq D(\rho \| \sigma) + \log C \quad \forall \text{ states } \rho$$

and

$$\text{tr} e^{\log \sigma + \sum_{k=1}^n \mathcal{E}_k^*(\log \omega_k)} \leq C \prod_{k=1}^n \|e^{\log \omega_k + q_k \log \sigma_k}\|_{1/q_k} \quad \forall \omega_k \succ 0$$

- ▶ Proof via Legendre: $D(\rho \| \sigma) = \sup_{\omega \succ 0} \{\text{tr} \rho \log \omega - \log \text{tr} e^{\log \omega + \log \sigma}\}$ [Petz]
- ▶ Not clear which side looks more intimidating...
- ▶ Useful choices: $\sigma_k = \mathcal{E}_k(\sigma)$ or $\sigma_k = I$, $\sigma = I$

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When specializing to $\sigma_k = I$, $\sigma = I$, recover equivalence between

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For example, can prove **uncertainty relations** via trace inequalities, as pioneered by Frank-Lieb:

- ▶ Maassen-Uffink: $S(X) + S(Z) \geq S(\rho) + 1$ via Golden-Thompson
- ▶ Six-state [Coles et al]: $S(X) + S(Y) + S(Z) \geq S(\rho) + 2$ via Lieb 3-matrix

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Applications and questions

- ▶ Can we prove new **uncertainty relations** involving multiple measurements (and even general quantum channels)? N -matrix GT?
- ▶ **Strong data-processing** inequalities fall into the framework:

$$D(\mathcal{E}(\rho) \parallel \mathcal{E}(\sigma)) \leq \eta D(\rho \parallel \sigma) \quad \forall \rho$$

- ▶ Tensorization holds classically, but fails quantumly:

$$(\mathcal{E}, C) \ \& \ (\mathcal{E}', C') \ \not\Rightarrow \ (\mathcal{E} \otimes \mathcal{E}', C \cdot C')$$

Examples include **non-additivity** of minimal output entropy. Useful?

- ▶ Computational complexity of testing validity of (families of) BL ineqs?
- ▶ Relation to works by Carlen-Maas?

Back to geometry. . .

Recall the **classical** Brascamp-Lieb inequalities in the **geometric case**:

$$\sum_{k=1}^n q_k S(P_k X) \geq S(X) \quad \forall \text{ RV } X \text{ on } \mathbb{R}^m$$

with P_k projections onto subspaces $V_k \subseteq \mathbb{R}^m$ s.th. $\sum_{k=1}^n q_k P_k = I_m$.

How can we formulate a **quantum version**? For any subspace $V \subseteq \mathbb{R}^m$,

$$L^2(\mathbb{R}^m) = L^2(V \oplus V^\perp) = L^2(V) \otimes L^2(V^\perp)$$

hence can define **reduced state** ρ_V for any state ρ on $L^2(\mathbb{R}^m)$.

This generalizes the usual partial trace. In general, can interpret as state of subset of modes after subjecting ρ to network of beamsplitters.

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Result: Geometric Quantum Brascamp-Lieb Inequality

Theorem

Let P_k projections onto subspaces $V_k \subseteq \mathbb{R}^m$ s.th. $\sum_{k=1}^n q_k P_k = I_m$. Then, for all states ρ on $L^2(\mathbb{R}^m)$ with finite first and second moments:

$$\sum_{k=1}^n q_k S(\rho_{V_k}) \geq S(\rho)$$

- ▶ For coordinate subspaces recover quantum Shearer inequality. [Carlen-Lieb]
- ▶ But already nontrivial for “Mercedes star” configuration in \mathbb{R}^2 :



- ▶ Can generate more ineqs. via Gaussian unitaries: $\text{Sp}_{2n} \hookrightarrow L^2(\mathbb{R}^m)$.
- ▶ Can also prove version conditioned on side information.

Sketch of proof

$$\sum_{k=1}^n q_k S(\rho_{V_k}) \geq S(\rho)$$

Implement classical proof strategy of Carlen–Cordero-Erausquin using **quantum heat flow** of König-Smith:

cf. [De Palma–Trevisan]

$$\frac{d}{dt} \rho = - \sum_{j=1}^m [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]]$$

Asymptotic scaling of entropy: $S(\rho_V(t)) \sim \dim V \log t$

- ▶ Inequality holds at $t = \infty$ if $\sum_k q_k \dim V_k \geq m$.

Quantum de Bruijn identity: $\frac{d}{dt} S(\rho) = J(\rho)$, a Fisher information.

- ▶ Can prove reverse inequality for Fisher information if $\sum_k q_k P_k \leq m$:

$$\sum_{k=1}^n q_k J(\rho_{V_k}) \leq J(\rho)$$

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Outlook



Duality between quantum relative entropy inequalities and trace inequalities. **Unifying framework** to tackle information theoretic questions. New family of **geometric** quantum Brascamp-Lieb inequalities.

Many exciting directions:

- ▶ Uncertainty relations from n -matrix GT?
- ▶ General structure of Gaussian BL inequalities?
- ▶ Sufficient conditions for tensorization?
- ▶ Applications of new trace inequalities?
- ▶ ...

Thank you for your attention!