Entanglement renormalization and CFT:
Quantum circuits for the Dirac field in $1+1$ dimensions

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It from Qubit workshop, Kyoto, June 2019

with Freek Witteveen, Volkher Scholz, Brian Swingle (1905.08821)
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Complexity of many-body quantum physics

Many-body states have **exponentially large** description:

\[ |\Psi\rangle = \sum_{i_1,\ldots,i_n} \Psi_{i_1,\ldots,i_n} |i_1,\ldots,i_n\rangle \]

In practice, entanglement local \(\sim\) compact description:

Start with local entangled pairs...

... and glue by applying local transformations:
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Start with local \textit{entangled pairs}…

\[ \ldots \text{and glue by applying} \textit{local transformations:} \]

\[ \ldots \]
Tensor networks in practice and theory

**Tensor network**: many-body state defined by contracting network of (local) tensors

- PEPS [Verstraete-Cirac]
- MERA [Vidal]
- Matrix product state

Numerical tool: **ansatz classes for many-body states**

- Geometrize entanglement: *area (RT) laws*
- Some are *quantum circuits*

Powerful *theoretical formalism* that provides ‘dual’ or ‘holographic’ descriptions of complex phenomena: topological order, . . .
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![PEPS](image) [Verstraete-Cirac]

![MERA](image) [Vidal]

matrix product state

[White, Fannes-Nachtergaele-Werner, Östlund-Rommer]

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Tensor networks and quantum field theory

Quantum field theories are defined in the continuum, while tensor networks are discrete and finitary. How to reconcile?

Two successful approaches:

- modify ansatz \(\mapsto\) continuum tensor networks
- relate discrete networks to correlation functions of continuum theory = unified perspective!

In either case...

What do tensor networks capture? Can we identify general construction principles? Why do tensor networks work well?

cf. plethora of rigorous results on gapped lattice systems in 1+1d [Hastings, ...]
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More generally: **How to study QI in QFT?**

In either case...

.getElements

- notions like subsystem, entropy, approximation, etc. subtle
- exciting recent progress. large body of work in mathematical physics.

cf. plethora of rigorous results on gapped lattice systems in 1+1d \[ [\text{Hastings, . . .}] \]

More generally:

- c-theorem from subadditivity, Bekenstein bound via relative entropy, renormalization vs QEC, approximate QEC, . . .
Our contributions

Result

We construct tensor networks for the Dirac CFT in 1+1 dimensions.

Key features:

▶ explicit construction – no variational optimization
▶ rigorous approximation of correlation functions
▶ quantum circuits that renormalize entanglement

We achieve this using tools from signal processing: multiresolution analysis and discrete/continuum duality from wavelet theory.

Also obtain sub-circuits for Majorana and Ising CFT. In prior work, we constructed (branching) MERA for critical free-fermion lattice models.
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Approach: Multi-scale Entanglement Renormalization Ansatz (MERA)

Tensor network ansatz for **critical systems**: [Vidal]

- introduce entanglement at all scales / disentangle & coarse-grain
- noise-resilient on quantum computer [Kim et al]
- reminiscent of holography [Swingle], starting point for tensor network models [Qi, HaPPY, Hayden-...-W]

Important to understand design principles! Can we bridge numerics and toy models?
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Tensor network ansatz for critical systems: [Vidal]

|$0\rangle \otimes N$

↓ local quantum circuit that prepares state from $|0\rangle \otimes N$

↑ entanglement renormalization

↓ organise q. information by scale

▶ introduce entanglement at all scales / disentangle & coarse-grain

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Result: Entanglement renormalization for Dirac CFT

Massless Dirac fermion in 1+1d: \[ i\gamma^\mu \partial_\mu \psi = 0 \]

We construct MERAs that target vacuum correlation functions \( C = \langle O_1 \cdots O_n \rangle \) of smeared observables.

Result (simplified)

Goodness depends on smearing, \#layers, quality parameter. Comes with 'dictionary' for mapping observables. Symmetries approximately inherited!

First rigorous proof that entanglement renormalization can work for a CFT!
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Two-point functions: A-priori error

For different values of quality parameter and number of layers:

<table>
<thead>
<tr>
<th>Quality (depth per layer)</th>
<th>10</th>
<th>3</th>
<th>10</th>
<th>2</th>
<th>10</th>
<th>1</th>
<th>10</th>
<th>0</th>
<th>10</th>
<th>1</th>
<th>10</th>
<th>2</th>
<th>10</th>
<th>3</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of layers</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td></td>
<td>2</td>
<td>4</td>
<td>6</td>
<td></td>
<td>8</td>
<td>10</td>
<td>2</td>
<td>4</td>
<td>6</td>
</tr>
</tbody>
</table>

Similarly for higher-point functions.
Two-point functions: Numerics

For different values of quality parameter and *large* number of layers:

\[ \langle T(x) T(y) \rangle \]
Verifying conformal data

▶ central charge: $S(R) = \frac{c}{3} \log R + c'$

▶ usual procedure: identify fields by searching for operators that coarse-grain to themselves

↔ diagonalize ‘scaling superoperator’ [Evenbly-Vidal]

▶ not necessary in order to compute correlation functions using our MERAs – theorem provides dictionary!
Verifying conformal data

- central charge: \( S(R) = \frac{c}{3} \log R + c' \)

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\[ \begin{array}{c|c|c}
K=L=1 & K=L=2 \\
\hline
\Delta_I & 0 & 0 \\
\Delta_\eta & 0.5 & 0.5 \\
\Delta_{\bar{\eta}} & 0.5 & 0.5 \\
\Delta_\varepsilon & 1 & 1 \\
\Delta_\sigma & 0.097 & 0.131 \\
\Delta_\mu & 0.170 & 0.120 \\
\end{array} \]

(for Majorana subtheory)

\[ \begin{array}{c}
\text{entropy} \\
\text{subsystem size} \\
\end{array} \]

\[ \begin{array}{c}
\frac{1}{2} \log(R) + 0.7 \\
\frac{2}{3} \log(R) + 0.73 \\
\end{array} \]

\( \times \) \( K=L=1 \)

\( + \) \( K=L=3 \)

\( \sim \sim \) diagonalize ‘scaling superoperator’ [Evenbly-Vidal]

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\[
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\]
\[
K = L = 3
\]

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\sim\text{ diagonalize ‘scaling superoperator’} \quad \text{[Evenbly-Vidal]}

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How does it work?

Key technique: Entanglement renormalization using wavelet theory.

Tool from signal processing to resolve signal by scale:

Mathematically, basis transform built from scalings and translates of single localized ‘wavelet’.

▶ second quantization: quantum circuit!

We construct wavelets that target Fermi sea of Dirac field using recent signal processing results (Selesnick’s Hilbert pairs).
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In more detail: Wavelets and MERA

Wavelet bases are built from scalings & translates of single wave packet:

\[ j = -1 \quad j = 0 \quad j = 1 \]

We can recursively resolve signal into different scales (multiresolution analysis):
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\[ \frac{1}{2} + \frac{1}{2} \]

Transform is implemented by circuit on single-particle Hilbert space:

- Discrete circuit resolves continuous signal by scale!
- Second quantization yields Gaussian MERA layer. [Evenbly-White]
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\[ = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} \]

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We can recursively resolve signal into different scales (multiresolution analysis):

\[ \text{scaling basis (scale } \geq j) = \frac{1}{2} \quad + \quad \frac{1}{2} \quad \text{scaling basis (scale } \geq j+1) \]

\[ \text{wavelet basis (scale } = j) \]

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\[
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Wavelets for the Dirac fermion

Massless Dirac equation in 1+1d:

\[
\begin{bmatrix}
\psi_1(x, t) \\
\psi_2(x, t)
\end{bmatrix} = \begin{bmatrix}
\chi_+(x - t) + \chi_-(x + t) \\
i\chi_+(x - t) - i\chi_-(x + t)
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\]

Need wavelets that target negative/positive momenta. Studied in signal processing, motivated by *directionality* and *shift-invariance*!

[Selesnick]

After second quantizing and careful analysis, obtain tensor network with rigorous approximation guarantees...
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Technical Result

Parameters:
- \( \mathcal{L} \) – number of layers
- \( \varepsilon \) – accuracy of wavelet pair
- \( \Gamma \) – support and smoothness of smearing functions

Consider correlation function with smeared fields & normal-ordered bilinears:

\[
C := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) O_1 \cdots O_M \rangle
\]

Theorem (simplified)

\[
\left| C_{\text{exact}} - C_{\text{MERA}} \right| \leq \Gamma \max\{2^{-\mathcal{L}/3}, \varepsilon \log \frac{1}{\varepsilon}\}
\]

We provide dictionary for \( C_{\text{MERA}} \) (discretize smearing functions in scaling basis etc).
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Dirac fermion on circle

Construction can be easily adapted to Dirac fermion on circle:

- finite number of layers once UV cut-off fixed
- systematic construction by \((\text{anti})\text{periodizing} \) wavelets
Summary and outlook

▶ entanglement renormalization quantum circuits for 1+1d Dirac CFT
▶ systematic construction with rigorous guarantees

Outlook:
▶ thermofield double, Dirac cones, ...
▶ building block for more interesting CFTs? starting point for perturbation theory or variational optimization?
▶ lift wavelet theory to quantum circuits! tensor network bootstrap?

Thank you for your attention!
How to build an approximate Hilbert pair  

Wavelets are built from filters $g[n]$ that relate functions at different scale:

$$\phi_{j-1}(x) = \sum_{n \in \mathbb{Z}} g[n] \phi_j(x - 2^{-j} n)$$

Necessary and sufficient to obtain orthonormal basis (roughly speaking):

$$|G(\theta)|^2 + |G(\theta + \pi)|^2 = 2, \quad G(0) = \sqrt{2}$$

Wavelets are related by Hilbert transform iff filters related by half-shift:

$$G(\theta) = H(\theta)e^{-i\theta/2}$$

To achieve this, find explicit approximation

$$e^{-i\theta/2} \approx e^{-iL\theta} \frac{D(-\theta)}{D(\theta)}.$$

Then, $H(\theta) = Q(\theta)D(\theta)$ and $G(\theta) = Q(\theta)e^{-iL\theta}D(-\theta)$ are approximately related by half-shift for any choice of $Q(\theta)$. 
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Non-relativistic 2D fermions – Lattice model

When put on lattice, massless Dirac fermion becomes: (Kogut-Susskind)

\[ H_{1D} \approx - \sum_n a_n^\dagger a_{n+1} + h.c. \]

Non-relativistic fermions hopping on 2D square lattice at half filling:

\[ H_{2D} = - \sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c. \]

Fermi surface:

- violation of area law: \( S(R) \sim R \log R \) (Wolf, Gioev-Klich, Swingle)
- Green’s function factorizes w.r.t. rotated axes
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Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

\[ W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \sim (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh} \]

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):

\[ \sim \text{approximation theorem} \text{ (with Haegeman, Swingle, Cotler, Evenbly, Scholz).} \]
Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

\[ W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \quad \sim \quad (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh} \]

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):

\[ \sim \text{approximation theorem} \quad \text{(with Haegeman, Swingle, Cotler, Evenbly, Scholz).} \]