Tensor scaling, quantum marginals, and moment polytopes

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based on joint work with Peter Bürgisser, Cole Franks, Ankit Garg, Rafael Oliveira, Avi Wigderson (ITCS’18, FOCS’18, FOCS’19)
Overview: Scaling and marginal problems

Interesting class of problems — with applications in q. information, algebra, analysis, computer science — that surprisingly can be phrased as optimization problems over noncommutative groups.

Null cone & moment polytopes $\leftrightarrow$ Norm minimization

(Geometric invariant theory) $\leftrightarrow$ (Optimization theory)

Result: General framework and effective algorithms.

Plan: Overview and illustration via tensor scaling problem.
Example: Matrix scaling

Let $X$ be a matrix with nonnegative entries. A scaling of $X$ is a matrix

$$Y = \begin{pmatrix} a_1 & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & a_n \end{pmatrix} X \begin{pmatrix} b_1 & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & b_n \end{pmatrix}$$

$(a_1,\ldots,b_n > 0)$.

A matrix is called doubly stochastic (d.s.) if row & column sums are 1.

Matrix scaling (Geometry): Given $X$, $\exists$ (approximately) d.s. scalings?

Permanent (Invariant Theory): ...iff per$(X) > 0$!

- can be decided in polynomial time
- find scalings by alternatingly fixing rows & columns
- convergence controlled by permanent

Connections to complexity, combinatorics, geometry, numerics, ...
Example: Matrix scaling

Let $X$ be a matrix with nonnegative entries. A scaling of $X$ is a matrix

$$
Y = \begin{pmatrix}
    a_1 \\
    \vdots \\
    a_n
\end{pmatrix}
\begin{pmatrix}
    b_1 \\
    \vdots \\
    b_n
\end{pmatrix}
\begin{pmatrix}
    \cdot & \cdots & \cdot \\
    \cdot & \cdots & \cdot \\
    \cdot & \cdots & \cdot \\
\end{pmatrix}
\begin{pmatrix}
    \cdot & \cdots & \cdot \\
    \cdot & \cdots & \cdot \\
    \cdot & \cdots & \cdot \\
\end{pmatrix}
\begin{pmatrix}
    \cdot & \cdots & \cdot \\
    \cdot & \cdots & \cdot \\
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$$

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[Sinkhorn]

[Linial et al]

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Further examples

- **Horn problem**  
  ∃ Hermitian matrices $A + B = C$ with spectrum $\alpha, \beta, \gamma$? [Franks]

- **Positivity of Littlewood–Richardson coefficients** [Knutson-Tao]

- **Operator scaling** [Gurvits, Garg et al, Ivanyos et al]

- **Non-commutative polynomial identity testing**

- **Validity of Brascamp–Lieb inequalities** [Bennett et al, Garg et al]

- **Solution of Paulsen problem** [Kwok et al]

All these are special cases of a general class of problems. Let us focus on ‘representative’ example involving tensors...
Quantum states and marginals

Global quantum state of $d$ particles is described by unit-norm tensor

$$X \in \mathcal{V} = (\mathbb{C}^n)^\otimes d = \mathbb{C}^n \otimes \ldots \otimes \mathbb{C}^n$$

State of individual particles described by quantum marginals $\rho_1, \ldots, \rho_d$:

$$\rho_k = X_k X_k^*, \text{ where } X_k \text{ is } k\text{-th principal flattening of } X$$

Quantum marginal problem: Which $\rho_1, \ldots, \rho_d$ are consistent with a global state $X$?

Answer only depends on eigenvalues $\lambda_i$ of $\rho_i$:

(e.g., for $d = 2$: consistent iff same eigenvalues)
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Tensor scaling and moment polytopes

Scaling of $X$: Tensor of the form $Y = (A_1 \otimes \ldots \otimes A_d)X$.

Tensor scaling problem: Given $X$, which $\lambda_1, \ldots, \lambda_d$ are consistent with its scalings (and limits)?

- $\{(\lambda_1, \ldots, \lambda_d)\}$ convex moment polytopes
- Encode local info about entanglement
- Exp. large $V/H$-representations

We provide algorithmic solution!
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An Algorithm

Given $X$, want to find scaling $Y$ with desired marginals – whenever possible. For simplicity, uniform marginals $(\rho_i \propto I, \lambda_i \propto 1)$ and $d = 3$.

Algorithm: Start with $Y = X$. For $t = 1, \ldots, T$:
Compute marginals $\rho_1, \rho_2, \rho_3$ of $Y$. If $\varepsilon$-close to uniform, stop.
Otherwise, replace $Y$ by $(e^{-\delta \rho_1^o} \otimes e^{-\delta \rho_2^o} \otimes e^{-\delta \rho_3^o}) Y$. $X^o = \text{traceless part}$

Result

Algorithm finds $Y = (A_1 \otimes A_2 \otimes A_3) X$ with marginals $\varepsilon$-close to uniform within $T = \text{poly}(\frac{1}{\varepsilon}, \text{input size})$ steps.

- generalizes to arbitrary $\lambda_i$, $d > 3$, (anti)symmetric tensors, MPS, ...
- solve quantum marginal problem by using random $X$

cf. algorithm by Verstraete et al which we analyzed in prior work
Why does it work?

“Otherwise, replace \( Y \) by \((e^{-\delta \rho_1} \otimes e^{-\delta \rho_2} \otimes e^{-\delta \rho_3})Y\).”

This step implements gradient descent for logarithm of

\[
N(A_1, A_2, A_3) = \| (A_1 \otimes A_2 \otimes A_3) X \|
\]

where \( A_1, A_2, A_3 \) have det=1. Indeed:

- geodesic gradient can be identified with \((\rho_1^o, \rho_2^o, \rho_3^o)\)!
- vanishes iff marginals uniform 😊
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- geodesic gradient can be identified with $(\rho_1^o, \rho_2^o, \rho_3^o)$!
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Non-commutative duality

For $N(g) = \|g \cdot X\|$, the following optimization problems are equivalent:

$$\inf_{g \in G} N(g) > 0 \iff \inf_{g \in G} \| \nabla \log N(g) \| = 0$$

[Kempf-Ness]

• primal: norm minimization, dual: scaling problem
• non-commutative version of LP duality
• equivalent to semistability of $X$

We develop quantitative duality theory and 1st & 2nd order methods.

All examples from introduction fall into this framework. Numerical algorithms that solve algebraic problems!

Everything works for general actions of reductive $G$. Norm is log-convex along geodesics.
Analysis of Algorithm

“Unless $\varepsilon$-close to uniform, replace $Y$ by $(e^{-\delta \rho_1} \otimes e^{-\delta \rho_2} \otimes e^{-\delta \rho_3}) Y$.”

To obtain rigorous algorithm, show:

- **progress in each step:** $\|Y_{\text{new}}\| \leq (1 - c_1 \varepsilon) \|Y\|
- **a priori lower bound:** $\inf_{\det=1} \| (A_1 \otimes A_2 \otimes A_3 ) X\| \geq c_2$

Then, $(1 - c_1 \varepsilon)^T \geq c_2$ bounds the number of steps $T$.

The first point follows from geodesic convexity estimates.

For the second, construct ‘explicit’ invariants with ‘small’ coefficients so that $P(X) \neq 0$ implies bound in terms of bitsize of $X$. 
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Summary and outlook

Effective algorithms for null cone and moment polytope problems, with applications incl. quantum marginal and tensor scaling problems. Based on geometric invariant theory and g-convex optimization.

Many exciting directions:

- Numerical studies in q. many-body systems or chemistry
- Quantum algorithms?
- Algorithms for other problems with natural symmetries?
- What are the ‘tractable’ problems in invariant theory? C~ F?

Thank you for your attention!
A general equivalence \( \mathcal{V} \subseteq \mathbb{P}(\mathcal{V}) \)

All points in \( \Delta(\mathcal{V}) \) can be described via invariant theory:

\[
\mathcal{V}_\lambda \subseteq \mathbb{C}[\mathcal{V}]_{(k)} \quad \Rightarrow \quad \frac{\lambda}{k} \in \Delta(\mathcal{V})
\]

(\( \lambda \) highest weight, \( k \) degree)

- Can also study multiplicities \( g(\lambda, k) := \# \mathcal{V}_\lambda \subseteq \mathbb{C}[\mathcal{V}]_{(k)}. \)
- This leads to interesting computational problems:

\[
\begin{align*}
g &= ? \\
g &> 0? \\
\exists s > 0 : g(s\lambda, sk) &> 0?
\end{align*}
\]

(\#-hard) (NP-hard) (our problem!)

Completely unlike Horn’s problem: Knutson–Tao saturation property does not hold, and hence we can hope for efficient algorithms!