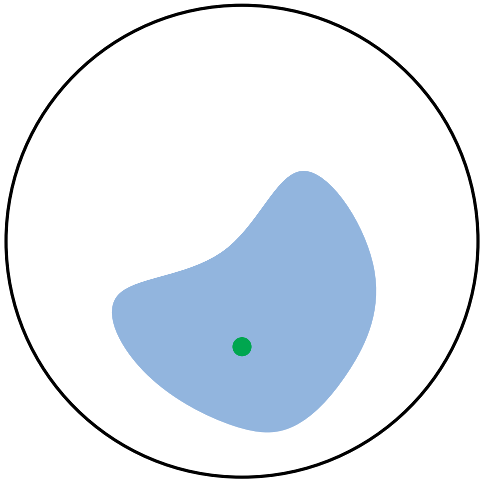

A Heisenberg Limit for Quantum Region Estimation

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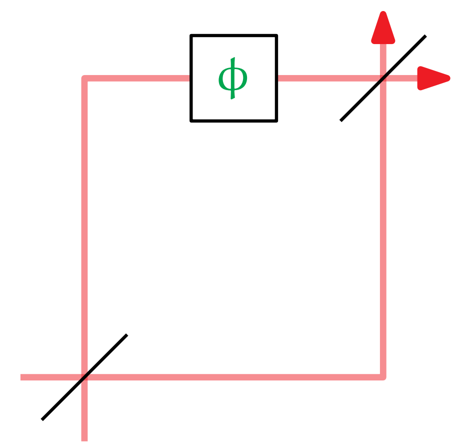
Quantum Parameter Estimation

x $\dashrightarrow$ ρ_B^x

unknown parameter — probe state

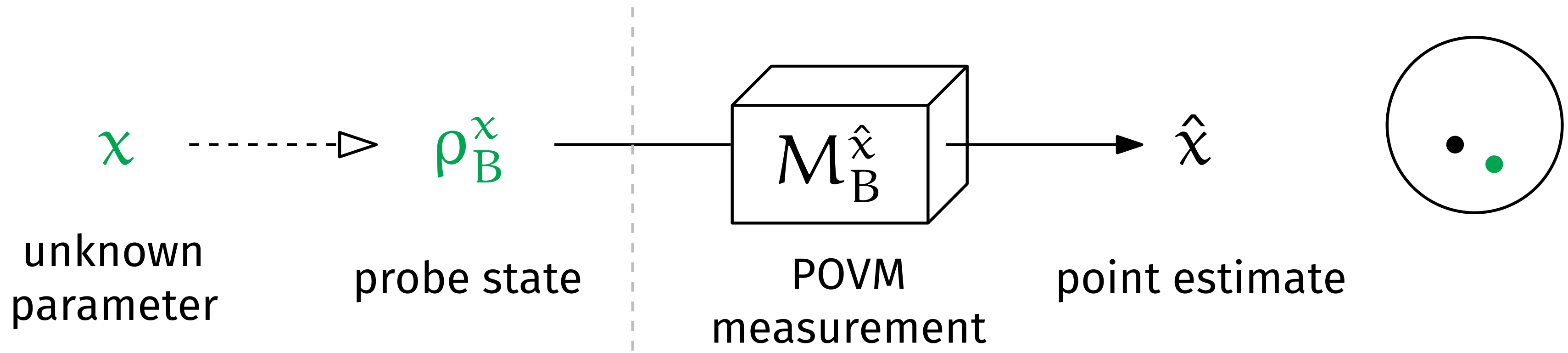
Goal: Determine parameter from probe state.

- high-precision measurements: frequencies, phases, positions, reference frames, ...
- quantum state tomography



This Talk: Fundamental Limits

Quantum Point Estimation



Cramér–Rao lower bound:

[Braunstein–Caves]

$$\text{m.s.e.} \equiv \langle (\hat{X} - x)^2 \rangle \geqslant \frac{1}{I(x)}$$

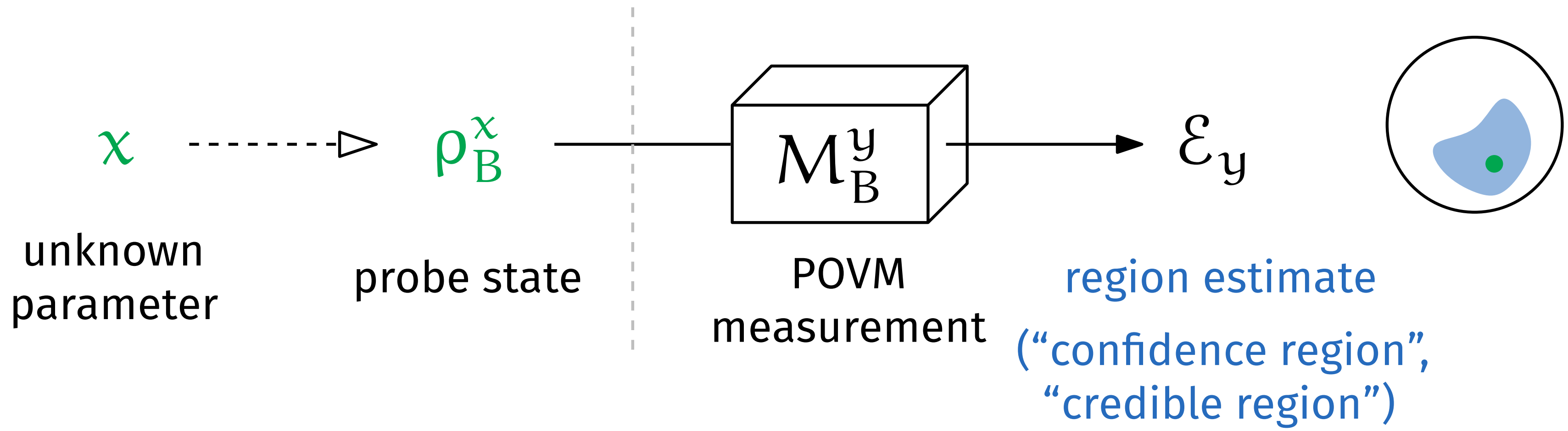
Fisher information

- $1/N$ scaling for i.i.d. probes
- can be overcome by using quantum entanglement [Giovanetti et al]

Van Trees inequality, Ziv–Zakai bounds, rate-distortion theory, … [Fraas, Tsang] [Yuen, Nair, Hall–Wiseman]

Quantum Region Estimation

[Blume–Kohout, Christandl–Renner, Ferrie, ...]



- Average success probability:

prior distribution

$$p_{\text{succ}} = \mathbb{P}(X \in \mathcal{E}_Y) = \sum_x p_x \sum_{y : x \in \mathcal{E}_y} \operatorname{tr} \rho_B^x M_B^y$$

- Maximal reported volume:

$$V_{\max} = \max_y |\mathcal{E}_y|$$

Parameter space can be continuous!

The Hypothesis Testing Lower Bound

Binary hypothesis testing:

POVM element to decide for ρ_0

$$\beta_\alpha(\rho_0, \rho_1) := \min \{ \operatorname{tr} \rho_1 E \ : \ 0 \leqslant E \leqslant \mathbb{1}, \operatorname{tr} \rho_0 E \geqslant \alpha \}$$

type II-error

significance

Theorem:

$$\boxed{\frac{V_{\mathrm{max}}}{|X|} \geqslant \sup_{\sigma_B} \beta_{p_{\mathrm{succ}}}\left(\rho_{XB}, \frac{\mathbb{1}_X}{|X|} \otimes \sigma_B\right)}$$

performance of region estimation

estimation scenario

- Independent of the estimator, “uncertainty relation”
- Linear cone program, can be evalutated for states of interest

Sketch of Proof

$$\rho_{XB} \longleftrightarrow \frac{\mathbb{1}_X}{|X|} \otimes \sigma_B$$

null hypothesis alternative hypothesis

Given region estimator with POVM (M_B^y), regions $(\mathcal{E}_y)$, construct binary hypothesis test

$$E_{XB} = \bigoplus_x |x\rangle\langle x| \otimes \sum_{y : x \in \mathcal{E}_y} M_B^y$$

- Significance:

$$\alpha = \operatorname{tr} \rho_{XB} E_{XB} = p_{\mathrm{succ}}$$

- Type-II error:

$$\beta = \operatorname{tr}\Big(\frac{\mathbb{1}_X}{|X|} \otimes \sigma_B\Big) E_{XB} = \cdots = \frac{V_{\mathrm{avg}}(\sigma_B)}{|X|} \leqslant \frac{V_{\mathrm{max}}}{|X|}$$

cf. converse for lossy joint source-channel coding [Kostina–Verdú]; [Hayashi–Tomamichel], [Vazquez–Vilar et al]; HT–CR duality

Variations on a Bound

- Average volume:

$$\boxed{\frac{V_{\text{avg}}}{|X|} \geqslant \beta_{p_{\text{succ}}}\left(\rho_{XB}, \frac{\mathbb{1}_X}{|X|} \otimes \rho_B\right)}$$

log is hypothesis-testing conditional entropy
(→ QIT, thermodynamics)

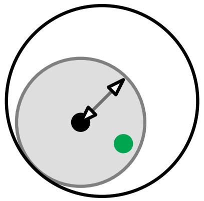
$$\Big\downarrow \text{ data processing}$$

$$h(p_{\text{succ}}) + p_{\text{succ}} \log V_{\text{avg}} + (1 - p_{\text{succ}}) \log |X| \geqslant H(X|B)$$

usual quantum conditional entropy

- Can obtain mean-square error bounds as corollaries!

$$\text{m.s.e.} \gtrsim e^{2H(X|B)}$$



treat point estimator as region estimator, optimize over radius

Covariant Estimation

Covariant Families

[Hayashi, Chiribella, ...]

$X = G$ compact Lie group, $\{U_g\}$ unitary representation on $\mathcal{H}_B$

$$\boxed{\rho_B^g = U_g \rho_B^0 U_g^\dagger}$$

initial probe state

- Phase estimation: $G = U(1) = \{e^{i\phi}\}$, $U_\phi = e^{iH\phi}$

General lower bound can be simplified $\rightsquigarrow$ one-shot version of "G-asymmetry":

$$\inf\{\|E_B^G\|_\infty : 0 \leqslant E_B \leqslant 1, \operatorname{tr} \rho_B^0 E_B \geqslant p_{\text{succ}}\}$$

averaged operator

Proof: Untwist; combine with chain rule for q. hypothesis testing. [Dupuis et al]

Universal Covariant Lower Bound

$$\mathcal{H}_B = \bigoplus_\lambda V_\lambda \otimes \mathbb{C}^{m_\lambda}$$

irreducible representation

multiplicity

Universal Lower Bound:

Any region estimator for an arbitrary covariant family on $\mathcal{H}_B$ and prior p_G satisfies:

$$\boxed{\frac{V_{\mathsf{max}}}{|G|} \geqslant \frac{\beta_{p_{\mathsf{succ}}}(p_G, \mathbb{1}_G/|G|)}{\sum_\lambda d_\lambda r_\lambda}}$$

where $d_\lambda := \dim V_\lambda$, $r_\lambda := \min\{d_\lambda, m_\lambda\}$. ← maximal Schmidt rank between irrep and multiplicity space

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Phase estimation: $J = \#$ of eigenvalues of H

where $d_\lambda := \dim V_\lambda$, $r_\lambda := \min\{d_\lambda, m_\lambda\}$. ← maximal Schmidt rank between irrep and multiplicity space

The General Heisenberg Limit

$$\mathcal{H}_B \rightsquigarrow \mathcal{H}_B^{\otimes N}$$
$$U_\phi \rightsquigarrow U_\phi^{\otimes N}$$

- entangled probes allowed!

Heisenberg Limit:

Any region estimator for an arbitrary covariant family on $\mathcal{H}_B^{\otimes N}$ and prior p_G satisfies:

$$\boxed{\frac{V_{\mathrm{max}}}{|G|} \geqslant \frac{\beta_{p_{\mathrm{succ}}}(p_G, \mathbb{1}_G/|G|)}{O(N^{\dim G})}}$$

Proof: Use Lie theory to show that $\sum_\lambda d_\lambda r_\lambda = O(N^{\dim G})$.

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Phase estimation: $J = O(N)$

Proof: Use Lie theory to show that $\sum_\lambda d_\lambda r_\lambda = O(N^{\dim G})$.

Example: State-Dependent Lower Bounds

Can also evaluate bounds for concrete families of probe states!

- GHZ: $\frac{1}{\sqrt{2}}(|0\dots0\rangle + |1\dots1\rangle)$, $H = \sigma_z$

  $J_{\text{eff}} \equiv 2 \rightsquigarrow$ const. lower bound (independent of N)!

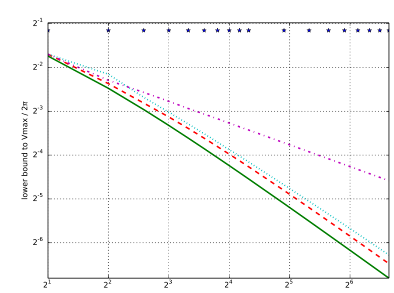
  Cramér–Rao predicts m.s.e. $\sim 1/(j_{max} - j_{min})^2 \sim 1/N^2$. [Hall–Wiseman]
  Local vs. global performance!

- Separable states: $|\psi_1\rangle \otimes \cdots \otimes |\psi_N\rangle$

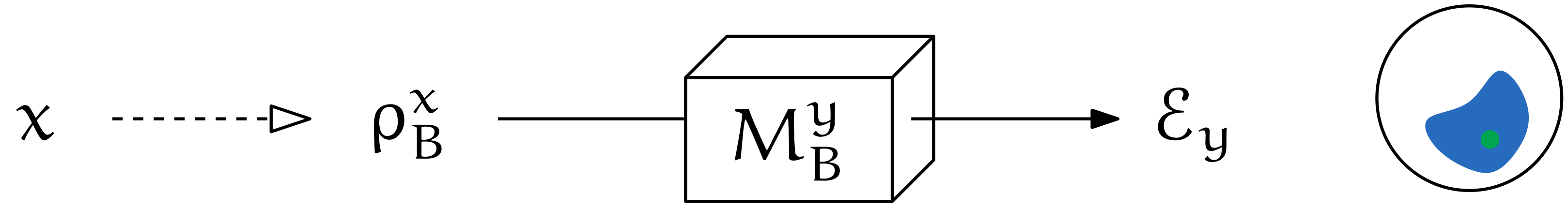
  $J_{\text{eff}} = O(\sqrt{N}) \rightsquigarrow$ standard quantum limit

- Energy-bounded probes: $\operatorname{tr} \rho_B^0 a^\dagger a \leqslant E$ cf. [Yuen, Nair]

  $J_{\text{eff}} = O(E)$



Method: Truncate state and count eigenspaces.



$$\frac{V_{\max}}{|X|} \geqslant \sup_{\sigma_B} \beta_{p_{\mathrm{succ}}}\left(\rho_{XB}, \frac{\mathbb{1}_X}{|X|} \otimes \sigma_B\right)$$

Thanks for your attention

$$\frac{V_{\max}}{|G|} \geqslant \frac{\beta_{p_{\mathrm{succ}}}(p_G, \mathbb{1}_G/|G|)}{\sum_\lambda d_\lambda r_\lambda} = \frac{\beta_{p_{\mathrm{succ}}}(p_G, \mathbb{1}_G/|G|)}{O(N^{\dim G})}$$