The holographic entropy cone

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joint work with S. Nezami, J. Sully (Stanford), N. Bao, H. Ooguri, B. Stoica (Caltech)
Holography: the gauge/gravity correspondence

boundary

d-dim QFT
large N, strongly coupled

bulk
(d+1)-dim q. gravity
semiclassical

AdS_{2+1}

CFT_{1+1}
Why care? (As a quantum information theorist...) 

Entanglement is “explained” via geometry, and vice versa. 

Holographic systems behave like q. error correcting codes. Further curious QIT properties (...)

QIT ideas useful to construct rigorous toy models of holography!
The holographic entropy formula

\[ S(A) = \min_{A' \sim A} |A'| \]

- entanglement entropy in boundary
- length of minimal geodesic in bulk
This talk

\[ S(A) = \min_{A' \sim A} |A'| \]

“Deconstruction” of the holographic entropy formula:

- Entropy cone
- Combinatorics
- Applications
The holographic entropy cone \( S(A) = \min_{A'} \) 

\[ \mathcal{C}_n = \{(S(A_1), \ldots, S(A_1A_2), \ldots, S(A_1 \cdots A_n)) \in \mathbb{R}^{2^n-1} \} \]

where we allow for arbitrary geometries and boundary regions.

This is a convex cone, the holographic entropy cone.

facets: entropy inequalities

extreme rays: most extreme entropy vectors

“Phase space” of holographic entropies.
A combinatorial version of holographic entropy

Instead of studying all possible smooth geometries, we may reduce to graphs:

Holographic entropy of subsystem = weight of min-cut in graph

Graphs to geometries via pair-of-pants decomposition → time slices of Lorentzian wormholes.
Application 1: Polyhedrality

**Theorem:** Each holographic entropy cone is polyhedral.

That is, there are:

- Finitely many entropy inequalities
- Finitely many extreme rays

Proof by reduction to graphs with a *fixed* number of vertices.

This is *not* known for general quantum states (and *false* for the Shannon entropy).
Application 2: Holographic entropy inequalities
Strong subadditivity in holography

\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]
Strong subadditivity in holography

\[ S(AB) + S(BC) = \]
Strong subadditivity in holography

\[ S(AB) + S(BC) \geq S(B) + S(ABC) \]

Making this precise requires finding decompositions of the minimal surfaces that work in all geometric configurations.

Of course, strong subadditivity for general quantum states is much more difficult [Lieb-Ruskai]. 12/20
# Proofs by contraction

\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

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"occurrence vectors"

**Theorem:** If table can be extended to Hamming contraction then the entropy inequality is valid.

Encodes cutting/pasting prescription. Easy to prove from graph model. Greedy algorithm.
Monogamy of mutual information

\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]

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\[ I(A : B) + I(A : C) \leq I(A : BC) \]

Does *not* hold for general quantum systems. It “excludes” non-trivial quantum Markov states, GHZ states, ...
A cyclic family of entropy inequalities

\[
\sum_{i=1}^{2k+1} S(A_i | A_{i+1} \ldots A_{i+k}) \geq S(A_1 \ldots A_{2k+1})
\]

Part of an infinite new family; unifies all previously known holographic entropy inequalities.

We could also verify entropy inequalities conjectured for the von Neumann entropy.

Proof by construction of explicit family of contractions.
Main application: Entropic constraints on the existence of smooth dual geometries.
The holographic entropy cones

$n \leq 4$ subsystems: subadditivity and monogamy are complete

$n \geq 5$ subsystems: several new inequalities

extreme rays have higher genus, nontrivial interior cycles
“Application” 3: Random tensor networks

Theorem: Holographic entropies are entropies of stabilizer states.

Probabilistic method: random tensor network states satisfy RT formula with high probability.

In particular, always quantum mechanical. Strong consistency check!

\[ S(A) = \min_{A' \sim A} |A'| \]

An extension of the method can be used to construct a toy model for the holographic correspondence.

[Hayden-Nezami-Qi-Thomas-W.-Zhao]

These are not graph states! Interestingly, stabilizers also have classical dual. Cf. [Ruskai et al], [Gross-W.]. 18/20
In particular, the RT formula really does behave like a von Neumann entropy. Conversely, can check conjectures about the latter via holography. Amusing aside: Kitaev-Preskill topological entropy (I_3) emerges as unique measure of area law violation.
Summary

Holographic entropy cone as organizing principle

Min-cuts in graphs as combinatorial model
• “proofs by contraction” of new entropy inequalities
• back to geometry: Lorentzian wormholes
• back to quantum states: random tensor networks

Thank you for your attention!