Kronecker coefficients and complexity theory

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Young diagrams

Young diagram $\lambda$:
- row lengths $\lambda_1 \geq \ldots \geq \lambda_m \geq 0$
- partition of $k$ into $\leq m$ parts

They parametrize the irreducible representations of:

Symmetric group $S_k$:
Specht module $[\lambda]$

General linear group $GL(m)$:
Weyl module $\bigvee_{\lambda}^m$
Well-known decompositions

Clebsch-Gordan rule for SU(2):

\[ V_i \otimes V_j = \bigoplus_{k=|i-j|}^{i+j} V_k \]

Schur-Weyl duality:

\[ \left( \mathbb{C}^m \right)^{\otimes k} = \bigoplus_{\lambda} \mathbb{V}_{\lambda}^m \otimes \mathbb{V}_{\lambda} \]

e.g., \[ \mathbb{V}_{\lambda} \] is the symmetric subspace
**Littlewood-Richardson coefficients**

\[ V^m_\lambda \otimes V^m_\nu = \bigoplus \bigodot \mathbf{c}^\lambda_\nu \downarrow V^m_\nu \]

**Littlewood-Richardson rule:**

\[ \mathbf{c}^\lambda_\nu = \# \text{ of LR tableaux of shape } \nu/\lambda \text{ with weight } \rho \]

**Honeycomb and hive models:** [Knutson-Tao]

\[ \mathbf{c}^\lambda_\nu = \# \text{ of honeycombs with boundary conditions} \\
= \# \text{ of integral hives with boundary conditions} \]

Both formulas count **combinatorial gadgets** – they are **evidently positive**!

Moreover, we can efficiently determine if nonzero. [Mulmuley-Sohoni]
Littlewood-Richardson coefficients

\[ V^m_\chi \otimes V^m_\rho = \bigoplus_v c^{\chi \rho}_{\nu} V^m_v \]

Saturation property: [Knutson-Tao]

\[ c^{s_\lambda s_\rho}_{s_\nu} > 0 \implies c^{\chi \rho}_{\nu} > 0 \]

Symplectic geometry: directly related to eigenvalues of Hermitian matrices with

\[ A + B = C \]

→ Horn’s inequalities
Kronecker coefficients

\[ [\lambda] \otimes [\mu] = \bigoplus \mathcal{g}_{\lambda\mu \nu} [\nu] \]

Many interesting connection to other areas of mathematics & applications (→later).
In part via:

\[ \text{Sym}^k (C^m \otimes C^m \otimes C^m) = \bigoplus \mathcal{g}_{\lambda\mu \nu} V^m \otimes V^m \otimes V^m \]

Despite 75+ years of history, many properties remain poorly understood!

Littlewood-Richardson coefficients are special Kronecker coefficients.
Kronecker coefficients: formulas

$$[\lambda] \otimes [\mu] = \bigoplus \chi_{\lambda \Box \mu} [\gamma]$$

Explicit formulas in various special cases:
- Two rows
- Hooks

[Orellana et al], [Blasiak-Mulmuley-Sohoni]
[Remmel], [Blasiak]

Recent progress on the Saxl conjecture:

$$g_{\Box g} > 0 \text{ whenever } g =$$

[Ikenmeyer], [Pak-Panova-Vallejo]

Open problem: Find combinatorial interpretation!
Kronecker coefficients: asymptotics

\[ G(m) = \{ (\lambda, \mu, \nu) : g_{\lambda \mu \nu} > 0 \} \]

Asymptotic support is convex cone: symplectic geometry [Mumford], [Kirwan]

outside: \( g \geq 0 \)  
inside: \( \exists s: g_{s \lambda, s \mu, s \nu} > 0 \)

in general, \( s > 1 \): failure of saturation, ”holes”!

\( g_{\lambda \mu \nu} \) is piecewise quasi-polynomial. [Meinrenken-Sjamaar]

Various other asymptotics have been studied:
Motivation I: The Kronecker polytopes

\[ \Delta(m) = \left\{ \frac{(\lambda, \mu, \nu)}{k} : g_{\lambda \mu \nu} > 0 \right\} \]

...is a convex polytope: the Kronecker polytope.

More generally: moment polytope associated with arbitrary representation of a compact connected Lie group.

- explicit inequalities known [Klyachko, Berenstein-Sjamaar, Ressayre, Vergne-W.]

- efficient algorithms of high interest in quantum physics: quantum marginal problem

Another example: Littlewood-Richardson coefficients give rise to Horn polytopes.
Motivation II: Geometric complexity theory

How many multiplications are required to multiply 2 x 2 matrices?

\[
\begin{bmatrix}
    a_{11} & a_{12} \\
    a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
\end{bmatrix}
= 
\begin{bmatrix}
    a_{11} \cdot b_{11} + a_{12} \cdot b_{21} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} \\
    \vdots & \vdots
\end{bmatrix}
\]

In fact, 7 < 8 are enough! $\Rightarrow O(n^{2.807\ldots})$ elementary multiplications for n x n matrices.

Best known algorithm: $O(n^{2.3729\ldots})$  

What is the minimal exponent of matrix multiplications?
Motivation II: Geometric complexity theory

Idea: Rephrase in terms of tensor varieties, study using algebraic geometry!

\[ v_{\text{hard}} \in \mathbb{M}_n \otimes \mathbb{M}_n \otimes \mathbb{M}_n^* \subseteq \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m \]

\[ v_{\text{easy}} = \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i \quad G = G(L(C^m))^\mathbb{C} \]

The goal is to show that: \[ v_{\text{hard}} \in G \cdot v_{\text{easy}} \]

This would imply that we need \( r \) elementary multiplications for \( n \times n \) matrices.

[Burgisser-Ikenmeyer]

Landsberg: \( r=7 \) is optimal for \( n=2 \). Similarly: Permanent vs. determinant (Valiant’s conjecture).
Representation-theoretic obstructions

Instead of determining equations for the varieties, we seek to find "representation-theoretic obstructions":

\[ V_\lambda \subseteq R(G \cdot v_{\text{hard}}) \quad \text{but} \quad V_\lambda \not\subseteq R(G \cdot v_{\text{easy}}) \]

This naturally leads to certain Kronecker coefficients and related multiplicities (symmetric Kronecker coefficients, plethysms, ...). E.g.:

\[
\begin{array}{c}
G \\
\end{array} \quad \begin{array}{c}
\lambda \\
\end{array}
\]

[Buergisser-Landsberg-Manivel-Weyman]

Much recent work on Kronecker coefficients has been motivated by this connection to geometric complexity theory.
Kronecker coefficients: mathematical challenges

\[ [\lambda] \otimes [\mu] = \bigoplus \mathcal{g}_{ \lambda \mu \nu} [\nu] \]

1. Decide when a Kronecker coefficient is non-zero!
   Asymptotic polytopes well-understood, but failure of saturation makes it “difficult”.

2. Find a positive, combinatorial formula!
   Like the Littlewood-Richardson rule.

3. Understand the failure of saturation!
   Minimal stretching factor? How to find holes?

This talk: Explicitly study the complexity of these problems!
Computational complexity primer
Computational complexity theory

Study of computational problems: decision problems ("is n a prime?") and counting problems ("how many prime factors does n have?").

Central question: What is the difficulty of a computational problem?

I.e., can we hope for an efficient solution? Or will all algorithms take a long time? Contrast with computability theory ("does there exist any algorithm") & algorithm engineering ("find a fast algorithm").
The complexity class P

**Problem instance** → **Algorithm** → **Answer**

*Input, encoded in bits*

**P**: Computational problems that admit an **efficient** algorithm. *i.e., runtime polynomial in the input size*

**Intuition**: Those are the computationally feasible problems.

**Examples**: Linear algebra; linear optimization; min-cut; Fourier transforms; ...  
*Often due to mathematical structure, dualities, ...*

We may then zoom in and ask for the most efficient algorithm & matching lower bounds. E.g., know how to multiply two $n$ by $n$ matrices in time $O(n^{2.372})$ [Le Gall], but best lower bound is $3n^2 - o(n^2)$ [Landsberg]!
The complexity class NP

Not all decision problems are known to admit an efficient algorithm. But often the answer can be efficiently verified! e.g., factoring a number vs. verifying a factorization; coloring a graph vs. checking a coloring

**NP:** If answer “YES” then there exists small certificate that can be efficiently verified.

Can be rather nontrivial to prove that a problem is in NP (e.g., UNKNOT). Many problems not in NP.
P vs. NP

**P:** There exists an efficient algorithm.

**NP:** If answer YES then there exists small certificate that can be efficiently verified.

**Conjecture:** \( P \neq NP \).

Widely believed to be true, for empirical as well as philosophical reasons:

“Surely, finding a proof must be harder than verifying it...”

Interestingly, there are proofs that exclude entire proof strategies of \( P \neq NP \)!
A glimpse at the complexity landscape

**P:** There exists an efficient algorithm.

**NP:** If answer YES then there exists small certificate that can be efficiently verified.

**CoNP:** If answer NO then there exists small certificate that can be efficiently verified.

Only a small part of the complexity landscape (time, space, random, quantum, ...).
Comparing complexity

X can be **reduced to** Y if X can be solved efficiently using an efficient algorithm for Y.

Y is **NP-hard** if any problem in NP can be reduced to Y.

Y is **NP-complete** if NP-hard and contained in NP.

**NP-complete problems exist!** [Cook], [Levin]

In fact, many natural combinatorial problems are NP-complete. [Karp]

If any NP-complete problem has an efficient solution, then P=NP.

Various possible definitions of reduction (reuse, post-processing, ...).
Complexity of *counting* problems

**P**: There exists an efficient algorithm.

**NP**: If answer YES then there exists small certificate that can be efficiently verified.

**#P**: Answer = number of certificates accepted by an NP-algorithm.  

[Valiant]

Natural complexity class for counting gadgets that are easily verified.

*e.g., counting 3-colorings of a graph, integral hives, ...*

Arguably what we would call a "positive, combinatorial formula"!  

[Mulmuley]
Complexity & representation theory
Branching problems as computational problems

\[ V = \bigoplus m_\lambda V_\lambda \]

- **Decision problem:** Decide if multiplicity > 0.
- **Counting problem:** Compute the multiplicity.

We may thus use computational complexity theory to study their difficulty!
Complexity of Littlewood-Richardson coefficients

\[ V^m_\lambda \otimes V^m_\mu = \bigoplus c^m_{\nu} V^m_\nu \]

Input: Three Young diagrams such that \(|\lambda| + |\mu| = |\nu|\)

- **Decision problem:** P
  
  Proof relies on honeycombs & LP results. [Mulmuley-Sohoni]

- **Counting problem:** \#P-complete
  
  Combinatorial formula shows that in \#P. Hardness by reduction from contingency tables. [Narayanan]

Thus *any* other \#P problem can be solved by computing LR coefficients! E.g., exists mapping \{graphs\} \rightarrow \{Young diagrams\} s.th. \# of 3-colorings = f(LR coeff).

Consequences largely unexplored...
Complexity of Kronecker coefficients

\[ [\lambda] \otimes [\mu] = \bigoplus \mathfrak{S}_{\mu \nu} [\nu] \]

Input: Three Young diagrams such that \(|\lambda| = |\mu| = |\nu|\)

- **Decision problem**: \( \text{NP-hard} \)  Is it in \( \text{NP} \)? [Ikenmeyer-Mulmuley-W.]

  This was previously conjectured to be in \( \text{P} \)!
  “Hopeless” to look for efficient algorithm (i.e., to find a simple characterization).

- **Counting problem**: \( \#\text{P-hard} \)  Is there a \( \#\text{P} \) formula?

  ...since LR coefficients are special Kronecker coefficients.

For Young diagrams of **bounded height**, both problems in \( \text{P} \)! [Christandl-Doran-W.]
Sketch of proof

**Theorem:** Deciding positivity of Kronecker coefficients is **NP-hard.**

Alternative characterization: \[ \# \bigwedge_{m} V_{\mu}^{m} \otimes V_{\nu}^{m} \subseteq \bigwedge^{n} (C^{m} \otimes \mathbb{C}) \]

Weight vectors = **point sets;** weight = **slice sums**

Deciding if there exists a point set with given slice sums is **NP-hard.** [Brunetti et al]

Relevant point sets are always “pyramids” \(\rightarrow\) correspond to **highest weight vectors.**
The failure of saturation

We are interested in finding examples of “holes”:

\[ g_{xy} = 0 \quad \text{but} \quad g_{xyz} > 0 \quad \text{for some} \quad s > 1 \]

Corollary: There exist “many” such holes and they can be constructed explicitly and efficiently.

Proof: We have a sequence of injective reductions

\[
\begin{align*}
&\quad \text{3D MATCHING} \quad \rightarrow \quad \text{4D PARTITION} \quad \rightarrow \quad \ldots \quad \rightarrow \quad \text{3D CONSISTENCY} \quad \rightarrow \quad \text{KRONECKER>0}
\end{align*}
\]

& 3D MATCHING has many “NO” instances.

The resulting holes are significantly beyond current methods – cannot even verify!
Asymptotic positivity

We may also consider the asymptotic positivity problem:
Given three Young diagrams,

$$\exists s: g_{xs} g_{ys} g_{sr} > 0 ?$$

That is, is the triple contained in the cone $C(m) ?$

**Theorem:** Deciding asymptotic positivity is in NP and CoNP.

- Suggests hardness of positivity problem is in part due to failure of saturation.
Sketch of proof

**Theorem:** Deciding asymptotic positivity is in NP and CoNP.

**NP:** Certificate is vector in $(\mathbb{C}^n)^{\otimes^3}$

Point in polytope can be computed efficiently. We prove that finite precision is not an issue (walls of polytope are not too steep).

**CoNP:** Certificate is separating hyperplane $\mathcal{H}_{1,2}$

Inequality can be verified efficiently (if also [Vergne-W.] given point at which to evaluate determinant polynomial).

Generalization to arbitrary groups, representations requires efficient algs for Lie algebra representation. 30/31
Summary

#P = “combinatorial formula” \hspace{1cm} \textit{NP-hardness of the positivity problem} \hspace{1cm} \text{explicit “holes”}

Complexity theory: conceptual framework for studying the difficulty of mathematical problems; a theory that can yield new mathematical results

New challenges in representation theory motivated by applications in geometric complexity theory, theoretical quantum physics
Thank you for your attention