Quantum circuits for the Dirac field in 1+1 dimensions

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based on arXiv:1903.xxxxx (with Scholz, Swingle, Witteveen) and arXiv:1707.06243 (with Haegeman, Swingle, Cotler, Evenbly, Scholz)
Tensor networks

\[ |\Psi\rangle = \sum_{i_1, \ldots, i_n} \Psi_{i_1, \ldots, i_n} |i_1, \ldots, i_n\rangle \]

Efficient variational classes for many-body quantum states:

- can have interpretation as quantum circuit

Useful theoretical formalism:

- geometrize entanglement structure: generalized area law
- bulk-boundary dualities: lift physics to the virtual level
- quantum phases, topological order, RG, holography, \ldots
MERA multi-scale entanglement renormalization ansatz (Vidal)

- self-similar layers that are short-depth quantum circuits
- variational class for critical systems in 1D
- interpretation: disentangle & coarse-grain
- network arises from tensor network renormalization:

↓ local quantum circuit that prepares state from $|0\rangle^\otimes N$

↑ entanglement renormalization

⇓ organize q. information by scale
MERA

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\[ \uparrow \text{entanglement renormalization} \]

\[ \downarrow \text{organize q. information by scale} \]
MERA and holography

- can always extend to ‘holographic’ mapping

- hyperbolic geometry (Swingle)
- starting point for tensor network models of holography (HaPPY; Hayden-...-W.)

- quantum error correction property = noise-resilience on QC (Kim et al)
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- quantum error correction property = noise-resilience on QC (Kim et al)
  ~ important to understand design principles
Tensor networks and quantum field theories

Tensor networks are discrete and finitary representations, while QFTs are infinite and defined in the continuum.

Two successful approaches:

- \textit{continuum} (cMPS, cMERA, \ldots)
- \textit{lattice} (MPS, PEPS, MERA, \ldots)

Questions:

- what do tensor networks capture?
- how to measure goodness of approximation?
- can we give rigorous construction principles?
- why do tensor networks work well?

\textit{cf. plethora of results on gapped 1D lattice systems in QIT/cond-mat}
Tensor networks for correlation functions

Given many-body system in state $\rho$ and choice of operators $\{O_\alpha\}$, define correlation function:

$$C(\alpha_1, \cdots, \alpha_n) = \text{tr}[\rho O_{\alpha_1} \cdots O_{\alpha_n}]$$

**Goal:** Design tensor network for correlation functions!

- unified perspective: system can be continuous – discreteness imposed by how we probe it
- tensor network for $\rho$ sufficient (if possible), but likely suboptimal

**Examples:** Zaletel-Mong (MPS/q. Hall states), König-Scholz (MPS/CFTs), cf. quantum marginal problem
Our results

We construct tensor networks for free fermion theories:

- **1D Dirac fermion** in continuum & lattice
- non-relativistic 2D fermions on lattice (Fermi surface)

Key features:

- tensor networks that target correlation functions
- **rigorous** approximation guarantees
- entanglement renormalization quantum circuits: (branching) MERA
- explicit construction, no variational optimization

We achieve this using tools from signal processing: wavelet theory.
1D Dirac fermion – Lattice result

Fermions hopping on infinite 1D lattice at half filling:

\[ H_{1D} = - \sum_n a^\dagger_n a_{n+1} + h.c. \]

- equivalent to ‘staggered’ massless Dirac fermions (Kogut-Susskind)
- easily solved using Fourier transform – but not using local q. circuit!

We construct MERA networks that target correlation functions:

\[ C(\{f_i\}) := \langle a^\dagger(f_1) \cdots a(f_{2N}) \rangle \]

Result (simplified)

\[ C_{\text{exact}} \approx C_{\text{MERA}} \]
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1D Dirac fermion – Numerics

Energy error

Green function $C(x, y) = \langle a_x^\dagger a_y \rangle$

bond dimension = $2^{\text{circuit depth}}$
1D Dirac fermion – Continuum result

Massless Dirac fermion in 1+1d:

\[ i \gamma^\mu \partial_\mu \psi = 0 \]

We construct circuits that target vacuum correlation functions in Dirac CFT:

\[ C({f_i, A_j}) := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \rangle \]

\( \Psi(f_i) \) smeared fields, \( A_j \) normal-ordered bilinears (e.g. smeared \( T, L_n \))

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Rigorous quantum circuit approximation for a QFT!
1D Dirac fermion – Verifying conformal data

- **central charge:** \( S(R) = \frac{c}{3} \log R + c' \)

- usual procedure: identify fields by searching for operators that coarse-grain to themselves

\[ \sim \] diagonalize ‘scaling superoperator’ (Evenbly-Vidal)

- in our case, no need to diagonalize – theorem contains ‘dictionary’
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1D Dirac fermion – Numerics

Two-point functions:

\[ \langle T(x) T(y) \rangle \]

\[ \Im \langle \Psi(x) \Psi(y) \rangle^\dagger \]

Similarly: OPE coefficients.
How to construct a free-fermion (= Gaussian) MERA?

Free-fermion ground states are Fermi seas filled with negative energy modes of single-particle Hamiltonian. This begs the question:

*How to perform entanglement renormalization on the single-particle level? Is there a single-particle variant of MERA?*

Yes – wavelet transforms!
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Yes – wavelet transforms!
Wavelets and renormalization

Fourier basis resolves signal into scales, but is completely nonlocal. In contrast, can also generate basis by scalings and translates of single localized wave packet – a wavelet:

$$j = -1 \quad j = 0 \quad j = 1$$

Then we can recursively resolve signal into different scales:

where

$$W_j = \text{span of wavelets at scale } j$$
$$V_j = \text{signals at scale up to } j = W_j \oplus W_{j+1} \oplus \ldots$$
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Wavelets and MERA

The basis transformation

$$V_j \rightarrow W_j \oplus W_{j+1}$$

is implemented by a *classical circuit* acting on *single-particle* Hilbert space:

Second quantization yields layer of a Gaussian MERA!

- in fact, obtain ‘holographic’ mapping (Qi)
- depth of classical circuit = depth of quantum circuit (Evenbly-White)

*Upshot:* To construct free-fermion ground state, design wavelet transform that targets positive/negative energy modes.
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1D Dirac fermion – Vacuum state

Massless Dirac equation in 1+1d:

\[ i \gamma^\mu \partial_\mu \psi = 0 \]

Negative energy modes:

- \( \chi_{\pm} \) supported on \( k < 0 \) / \( k > 0 \)
- \( \psi_{1,2} \) related by \( -i \text{sign}(k) \) at \( t = 0 \) (Hilbert transform)
- can choose any basis of Fermi sea…

Goal: Design pair of wavelets related by Hilbert transform!
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\[
\begin{bmatrix}
\psi_1(x, t) \\
\psi_2(x, t)
\end{bmatrix} = \begin{bmatrix}
\chi_+(x - t) + \chi_-(x + t) \\
i\chi_+(x - t) - i\chi_-(x + t)
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1D Dirac fermion – Hilbert wavelet pairs

Such wavelet pairs have been studied in the signal processing community:

- motived by directionality and shift-invariance (!)
- impossible exactly with local circuit, but possible to arbitrary accuracy (Selesnick)

After second quantizing and careful analysis, obtain tensor network with rigorous approximation guarantee...
Consider correlation function with smeared fields & normal-ordered bilinears:

\[
C(\{f_i, A_j\}) := \langle \Psi^\dagger(f_1) \cdots \Psi(f_{2N}) A_1 \cdots A_M \rangle
\]

Result (simplified)

\[
| C_{\text{exact}} - C_{\text{MERA}} | \leq \Gamma \max \{2^{-L/4}, \varepsilon \log \frac{1}{\varepsilon} \}
\]

In particular, all conformal symmetries approximately inherited.
1D Dirac fermion – Result

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Parameters:

- \( \mathcal{L} \) – number of layers
- \( \varepsilon \) – accuracy of Hilbert pair
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- \( L \) – number of layers
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1D Dirac fermion – Circle

Construction also works for Dirac fermion on circle:

- finite number of layers once UV cut-off fixed
- systematic construction by (anti)periodizing wavelets
- only top layers change – wavelet modes start ‘wrapping around’
Non-relativistic 2D fermions – Lattice model

\[ H_{1D} \cong - \sum_{n} a_{n}^\dagger a_{n+1} + h.c. \]

Non-relativistic fermions hopping on 2D square lattice at half filling:

\[ H_{2D} = - \sum_{m,n} a_{m,n}^\dagger a_{m+1,n} + a_{m,n}^\dagger a_{m,n+1} + h.c. \]

Fermi surface:

- violation of area law: \( S(R) \sim R \log R \) (Wolf, Gioev-Klich, Swingle)
- Green’s function factorizes w.r.t. rotated axes
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Non-relativistic 2D fermions – Branching MERA

Natural construction – perform wavelet transforms in both directions:

\[ W\psi = \psi_{\text{low}} \oplus \psi_{\text{high}} \sim (W \otimes W)\psi = \psi_{ll} \oplus \psi_{lh} \oplus \psi_{hl} \oplus \psi_{hh} \]

After second quantization, obtain variant of branching MERA (Evenbly-Vidal):

Similar approximation theorem holds.
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Summary and outlook

- entanglement renormalization quantum circuits for free fermions
- explicit construction with rigorous guarantees (lattice + continuum)

Outlook:

- thermofield double, massive theories, Dirac cones, ...
- building block for more interesting CFTs? starting point for perturbation theory or variational optimization?

Thank you for your attention!