Playing Games with Quantum Bits

Michael Walter

Leve de wiskunde! 2021
Quiz 1: Which of these are quantum computers?
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A

B

C

D
Quiz 2: In what ways does a quantum computer differ from a regular one?

A. Much faster clock speed (GHz)
B. Much smaller in size
C. Much bigger memory
D. Many more calculations in parallel
E. It uses quantum mechanics
Quiz 2: In what ways does a quantum computer differ from a regular one?

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We’ll talk about its mathematics.
Quantum Information

A young research field that combines two scientific revolutions of the past century:

Quantum Mechanics + Computer Science

Idea: Let’s try to use the power of quantum mechanics to get...

- faster algorithms
- higher rates
- better security

Quantum Computing
Quantum Communication
Quantum Cryptography

...than what is possible in a classical world.
Classical vs Quantum

The goal of today’s lecture is to give you an impression of...

1. the **mathematics of quantum bits** (no physics required!)
2. its **strange** and **powerful** implications

To get started, let us consider a game...
A curious game

Three players play against referee. Referee sends “questions” \( x, y, z \) (bits). They respond with “answers” \( a, b, c \) (bits). They **cannot communicate**.

![Diagram of the game](image)

**winning condition**

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<th>( x )</th>
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Question: Is there a winning strategy?
A curious game

Three players play against referee. Referee sends “questions” \( x, y, z \) (bits). They respond with “answers” \( a, b, c \) (bits). They cannot communicate.

Can we find “answer functions” \( a(x), b(y), c(z) \) that work?
A curious game

Three players play against referee. Referee sends “questions” \( x, y, z \) (bits). They respond with “answers” \( a, b, c \) (bits). They cannot communicate.

Can we find “answer functions” \( a(x), b(y), c(z) \) that work? If yes...

\[
(a(0) + b(0) + c(0)) + (a(0) + b(1) + c(1)) + (a(1) + b(0) + c(1)) + (a(1) + b(1) + c(0))
\]

...would be odd. But every answer appears twice. **Contradiction!**
A curious game

Three players play against referee. Referee sends “questions” $x, y, z$ (bits). They respond with “answers” $a, b, c$ (bits). They cannot communicate.

This game cannot be won in a classical world. But it turns out that using quantum bits our three protagonists can find a perfect winning strategy!
From bits to quantum bits
Bits and Probability

Bit: 0 or 1

Probabilities: How to describe the outcome of a coin toss?

50% + 50%
Bits and Probability

Bit: 0 or 1

Probabilities: How to describe the outcome of a coin toss?

\[ p_0 + p_1 \]
Bits and Probability

**Bit:** 0 or 1

**Probabilities:** How to describe the outcome of a coin toss?

\[ p_0 \cdot 0 + p_1 \cdot 1 \]
Bits and Probability

Bit: \( |0\rangle \) or \( |1\rangle \)

Probabilities: How to describe the outcome of a coin toss?

\[ p_0 |0\rangle + p_1 |1\rangle \]

We write \( |\ldots\rangle \) to avoid confusing “0” and “1” with probabilities.
Bits and Probability

Bit: \(|0\rangle\) or \(|1\rangle\)

Probabilities: How to describe the outcome of a coin toss?

\[ p_0\, |0\rangle + p_1\, |1\rangle \]

\[ p_0 + p_1 = 1, \quad p_0 \geq 0, \quad p_1 \geq 0 \]
Bits and Probability

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Bits and Probability

Bit: \(|0\rangle\text{ or } |1\rangle\)

Probabilities: How to describe the outcome of a **coin** toss?

\[ p_0 \, |0\rangle \, + \, p_1 \, |1\rangle \]

\[ p_0 + p_1 = 1, \, p_0 \geq 0, \, p_1 \geq 0 \]

Operations: Can “flip” or “negate” a bit...

\[ \text{NOT } |0\rangle = |1\rangle \]

\[ \text{NOT } |1\rangle = |0\rangle \]
Bits and Probability

Bit: \( |0\rangle \) or \( |1\rangle \)

Probabilities: How to describe the outcome of a coin toss?

\[
\begin{pmatrix}
  p_0 \\
  p_1
\end{pmatrix}
\]

\[p_0 + p_1 = 1, \quad p_0 \geq 0, \quad p_1 \geq 0\]

Operations: Can “flip” or “negate” a bit...

\[\text{NOT } |0\rangle = |1\rangle\]

\[\text{NOT } |1\rangle = |0\rangle\]
Quantum bit ("qubits")

The state of a qubit is also described by two numbers:

\[ a_0 |0\rangle + a_1 |1\rangle \]

These are called "amplitudes", can be negative, and satisfy a new rule:

\[ a_0^2 + a_1^2 = 1 \]

For example: \( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \)

How strange! What can we do with quantum bits?
Measuring a quantum bit

Quiz: What state do we need to imitate fair coin toss? More than 1 option?

Answer:

These are called the “plus” and “minus” states.
Quantum operations

Bit flip:

\begin{align*}
\text{NOT } |0\rangle &= |1\rangle \\
\text{NOT } |1\rangle &= |0\rangle
\end{align*}

same as before!

Hadamard operation:

\begin{align*}
H |0\rangle &= |+\rangle \\
H |1\rangle &= |-\rangle
\end{align*}

reflection along 22.5 degree axis

Quiz: How can we use these to create \( |-\rangle \) from \( |0\rangle \)?

Answer: First apply NOT and then \( H \)!

Quiz: How can we tell \( |+\rangle \) and \( |-\rangle \) apart?

Answer: Apply Hadamard and measure!
Quirky time!

www.quantum-quest.nl/ldw
Once more with mathematics

\[ |0\rangle \xrightarrow{H} \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \]

\[ H \quad \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle \right) = |0\rangle \]

As amplitudes can be negative, they can cancel. **Interference!**

Just like composers use it to create beautiful music, quantum computers use it to zoom into the **right answer.**
More than one quantum bit?

One **amplitude** for every possible bit string. E.g., for **three qubits**:

\[
\begin{align*}
a_{000} |000\rangle &+ a_{001} |001\rangle + \ldots + a_{111} |111\rangle
\end{align*}
\]

no nice way to visualize 😞

If we **measure** all qubits, get outcome **000** with probability \(a_{000}^2\) etc.

Can apply **operations** we already know on individual qubits:

\[
H_2 \ |?0\rangle = \frac{1}{\sqrt{2}} \ |?0\rangle + \frac{1}{\sqrt{2}} \ |?1\rangle
\]

There are many other operations.
How to win the curious game?

1. Before game begins, Alice, Bob, Charlie create the following state:

\[ \frac{1}{2} \left( |000\rangle - |011\rangle - |101\rangle - |110\rangle \right) \]

Each gets one of the three qubits.

2. During the game, they all act in the same way:

- **If question is 0**, measure qubit.
- **If question is 1**, apply Hadamard and then measure qubit.
Why does it work?

\[
\frac{1}{2} (|000\rangle - |011\rangle - |101\rangle - |110\rangle)
\]

<table>
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In the first case, Alice/Bob/Charlie simply measure their qubit → outcome is one of 000, 011, 101, 110 → sum is even

The other cases are more complicated. Can you see how it works? Homework! 😊

Quirky time!
What does it mean?

Quantum mechanics allows for strong correlations that are impossible otherwise. Our game shows this very clearly. This is called entanglement.

Einstein was unhappy and tried to get rid of it. He was wrong!

In our research group, we study the mathematics of entanglement. We want to understand what patterns there are and what to do with them.
Thanks for your attention!