Entanglement Polytopes

Multi-Particle Entanglement from Single-Particle Information

Michael Walter

joint work with Matthias Christandl, Brent Doran (ETH Zürich), and David Gross (Univ. Freiburg)
Multi-Particle Entanglement

How entangled is a given multi-particle quantum state prepared in the laboratory?
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What (if anything) can be said using *local tomography*? efficient!
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How entangled is a given multi-particle quantum state prepared in the laboratory?

What (if anything) can be said using local tomography?

globally pure

efficient!
Pure-State Entanglement

A pure state $\rho = |\psi\rangle\langle\psi|$ is entangled if and only if

\[ |\psi\rangle \neq |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle \]

Equivalent:

$\rho$ is unentangled iff all reduced density matrices $\rho_k$ are pure.
A pure state $\rho = |\psi\rangle\langle\psi|$ is entangled if and only if

$$|\psi\rangle \neq |\psi_1\rangle \otimes \ldots \otimes |\psi_N\rangle$$

Equivalent:

$\rho$ is unentangled iff all reduced density matrices $\rho_k$ are pure.

can verify using local tomography
Two Qubits

Schmidt decomposition

\[ |\psi\rangle = \sqrt{\lambda} |00\rangle + \sqrt{1 - \lambda} |11\rangle \]

\( (0.5 \leq \lambda \leq 1) \)
Two Qubits

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\[ |\psi\rangle = \sqrt{\lambda} |00\rangle + \sqrt{1 - \lambda} |11\rangle \]

maximal eigenvalue

\[ \rho_1, \rho_2 \sim \begin{pmatrix} \lambda & \varepsilon \\ \varepsilon & 1 - \lambda \end{pmatrix} \]

\( \mathbb{C}^2 \otimes \mathbb{C}^2 \)

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\[ |\psi\rangle = \sqrt{\lambda} |00\rangle + \sqrt{1-\lambda} |11\rangle \]

(0.5 \leq \lambda \leq 1)

maximal eigenvalue

\[ \rho_1, \rho_2 \sim \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} \]

Two classes

product states

entangled states

\[ \times \sqrt{0.5} (|00\rangle + |11\rangle) \]
Two Qubits

Schmidt decomposition

\[ |\psi\rangle = \sqrt{\lambda} |00\rangle + \sqrt{1 - \lambda} |11\rangle \]

(maximal eigenvalue)

\[ \rho_1, \rho_2 \sim \begin{pmatrix} \lambda & \sqrt{1 - \lambda} \\ \sqrt{1 - \lambda} & 1 - \lambda \end{pmatrix} \]

Two classes

product states

entangled states

\[ \text{can be converted into } \sqrt{0.5} (|00\rangle + |11\rangle) \text{ by local operations and post-selection (SLOCC)} \]
Eigenvalues of reduced density matrices characterize entanglement of global state.

Two Qubits

Schmidt decomposition

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(0.5 \leq \lambda \leq 1)

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entangled states

Eigenvalues of reduced density matrices characterize entanglement of global state.
Multi-Partite Systems

- **no** Schmidt decomposition
- rank of reduced density matrices **not** enough
- generically: **infinitely** many classes, labeled by \( \exp(N) \) many continuous parameters

\[ |\psi\rangle \]
Multi-Partite Systems

- no Schmidt decomposition
- rank of reduced density matrices not enough
- generically: infinitely many classes, labeled by \( \exp(N) \) many continuous parameters

\[
| \psi \rangle
\]

Eigenvalues of reduced density matrices can still give useful information!
Three Qubits

Six classes

\[ |GHZ\rangle = |000\rangle + |111\rangle \]
\[ |W\rangle = |100\rangle + |010\rangle + |001\rangle \]
\[ |B1\rangle = |0\rangle \otimes (|00\rangle + |11\rangle) \]
\[ |B2\rangle, |B3\rangle \]
\[ |000\rangle \]

Dür, Vidal & Cirac (2000)

Han, Zhang & Guo (2004)
Botero & Mitchison (p.c.)
Sawicki, W. & Kus (2012)
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maximal eigenvalues of \( \rho_1, \rho_2, \rho_3 \)

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\(|W\rangle = |100\rangle + |010\rangle + |001\rangle\)
\(|B1\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)\),
\(|B2\rangle, |B3\rangle\)
\(|000\rangle\)

Lower pyramid is witness for GHZ class!

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Entanglement Polytopes

\[ \Delta_C = \{ \vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \text{ for } \psi \in \mathcal{C} \} \]
**Entanglement Polytopes**

Our main results:

- convex polytope!
- finite hierarchy
- algorithm to compute using computational invariant theory (difficult)

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using results from Brion (1987), Kempf & Ness (1979)

algebraic geometry / GIT
Entanglement Polytopes

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using results from Brion (1987), Kempf & Ness (1979) algebraic geometry / GIT

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cf. Quantum Marginal Problem

Klyachko (2004)
Entanglement Criterion

- efficient, requires only linearly many measurements
- robust against small noise ($\psi \approx$ pure)
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$\vec{\lambda} \notin \Delta C \implies \psi \notin C$

“Bell inequalities”

violation of “Bell inequality”
Purity and Noise

Purity: \( p = \text{tr} \rho^2 \)
(can be estimated using two-body measurements)

**Fact:** If \( p \geq 1 - \varepsilon \) then there exists a pure state \( |\psi\rangle \) with \( \langle \psi | \rho | \psi \rangle \geq 1 - \varepsilon \) whose local eigenvalues differ by \( \lesssim N\varepsilon \).

*Impurity enlarges effective error bars!*
Thank you!

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http://www.itp.phys.ethz.ch/people/waltemic/polytopes

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