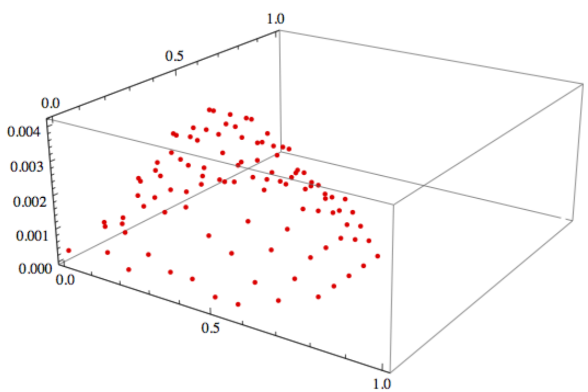
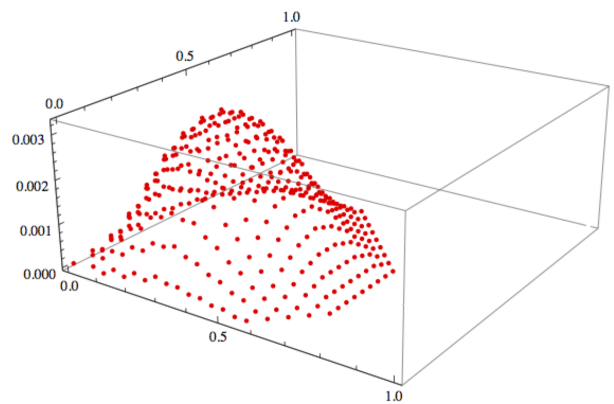
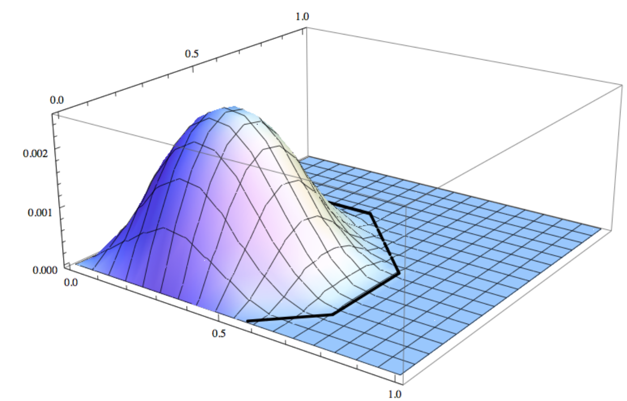
Quantum Marginals and Classical Moments

M. Walter (ETHZ)







joint work with M. Christandl, B. Doran, S. Kousidis

arXiv:1204.0741 and FOCS 2012

Plan for this Talk

1. Introduction to the Quantum Marginal Problem

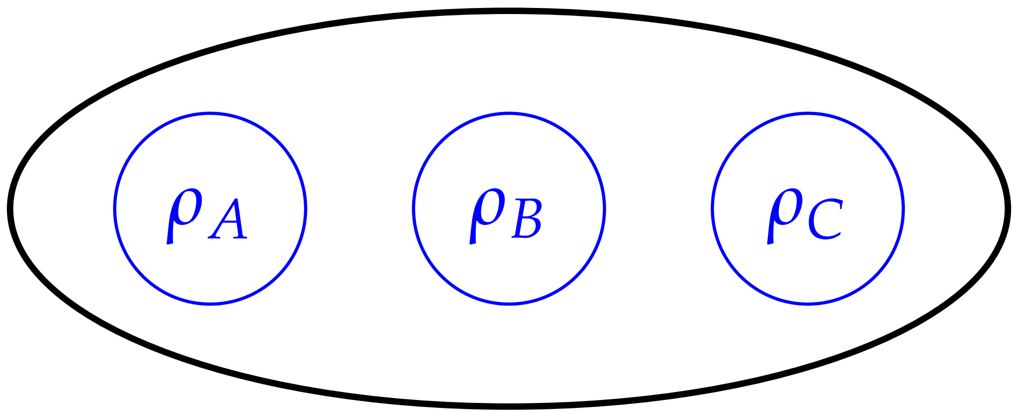
2. Marginals of Random States

3. The Branching Problem

4. Semiclassical Limit

One-Body Quantum Marginal Problem

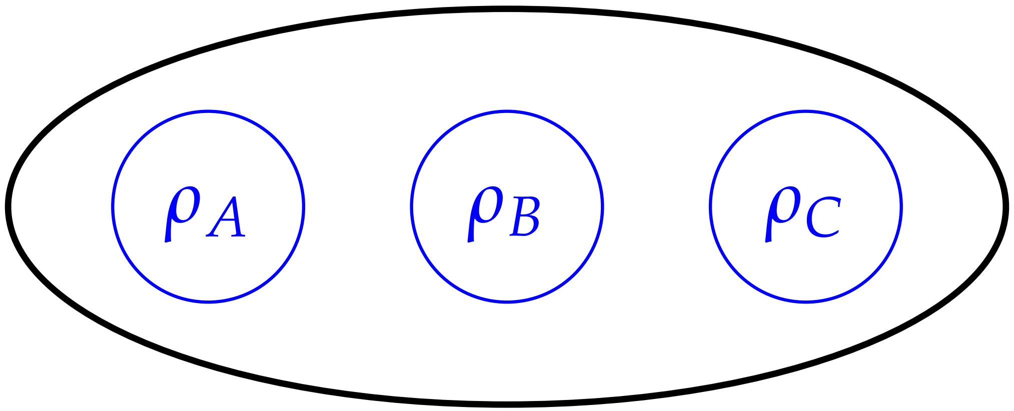
$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ pure state



Which collections ρ_A, ρ_B, ρ_C of reduced density matrices ("quantum marginals") are compatible?

One-Body Quantum Marginal Problem

$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ pure state



Which collections ρ_A, ρ_B, ρ_C of reduced density matrices ("quantum marginals") are compatible?

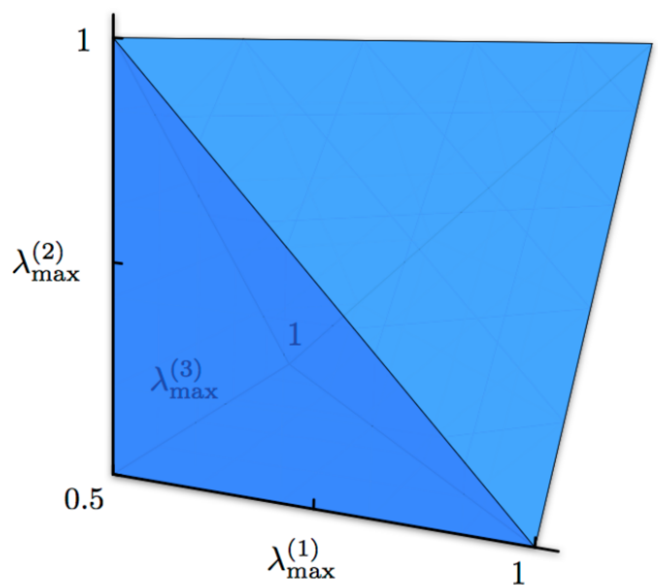
Only depends on local eigenvalues $\vec{\lambda}_A, \vec{\lambda}_B, \vec{\lambda}_C$!

Examples

- Bipartite Systems: $\mathbb{C}^d \otimes \mathbb{C}^d$

$$\rho_A, \rho_B \text{ compatible} \iff \vec{\lambda}_A = \vec{\lambda}_B$$

- Three Qubits: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

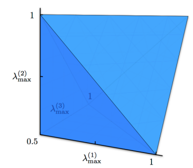


$$\lambda_{A,\max} + \lambda_{B,\max} \leq \lambda_{C,\max} + 1$$
$$\lambda_{A,\max} + \lambda_{C,\max} \leq \lambda_{B,\max} + 1$$
$$\lambda_{B,\max} + \lambda_{C,\max} \leq \lambda_{A,\max} + 1$$

General Features of Solution

$$\Delta = \left\{ (\vec{\lambda}_A, \vec{\lambda}_B, \vec{\lambda}_C) \text{ compatible} \right\}$$

- Convex polytope



Kirwan

- Explicit linear inequalities:

Klyachko

$$\sum_i a_{\pi(i)} \lambda_{A,i} + \sum_j b_{\tau(j)} \lambda_{B,j} \leq \sum_k c_{\sigma(k)} \lambda_{C,k}$$

whenever $[\pi]_a \otimes [\tau]_b \cap \iota^* [\sigma]_c \neq 0 \in H^*$.

algebraic geometry
Schubert calculus

- Representation theory

Christandl–Mitchison, …

$g_{\alpha,\beta,\gamma}$

Kronecker coefficients

How about the Diagonals?

$$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}| \text{ pure state}$$

$$\rho_A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ * & & a_d \end{pmatrix} \qquad \rho_B = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ * & & b_d \end{pmatrix} \qquad \rho_C = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ * & & c_d \end{pmatrix}$$

There are no constraints!

$$|\psi_{ABC}\rangle = \Big(\sum_i \sqrt{a_i}\,|i\rangle\Big) \otimes \Big(\sum_j \sqrt{b_j}\,|j\rangle\Big) \otimes \Big(\sum_k \sqrt{c_k}\,|k\rangle\Big)$$

No hope of solving the QMP in this way!?

2. Marginals of Random Quantum States

The “Random Marginal Problem”…

$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ Haar-random pure state

$\rho_A = (\cdots)$ $\rho_B = (\cdots)$ $\rho_C = (\cdots)$

What is the joint distribution of the marginals?

The "Random Marginal Problem"…

$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ Haar-random pure state

$\rho_A = (\,\cdots\,)$ $\rho_B = (\,\cdots\,)$ $\rho_C = (\,\cdots\,)$

What is the joint distribution of the marginals?

$\vec{\lambda}_A = (\lambda_{A,1}, \ldots)$ $\vec{\lambda}_B = (\lambda_{B,1}, \ldots)$ $\vec{\lambda}_C = (\lambda_{C,1}, \ldots)$

What is the joint distribution of the marginal eigenvalues?

Equivalent, since measure $U(d) \otimes U(d) \otimes U(d)$-invariant!

The "Random Marginal Problem"…

$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ Haar-random pure state

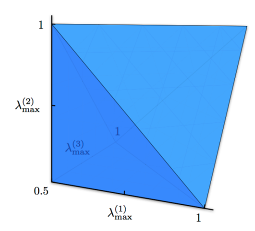
$\rho_A = (\cdots)$ $\rho_B = (\cdots)$ $\rho_C = (\cdots)$

What is the joint distribution of the marginals?

$\vec{\lambda}_A = (\lambda_{A,1}, \ldots)$ $\vec{\lambda}_B = (\lambda_{B,1}, \ldots)$ $\vec{\lambda}_C = (\lambda_{C,1}, \ldots)$

What is the joint distribution of the marginal eigenvalues?

This is a probability measure on the marginal polytope.



Strategy

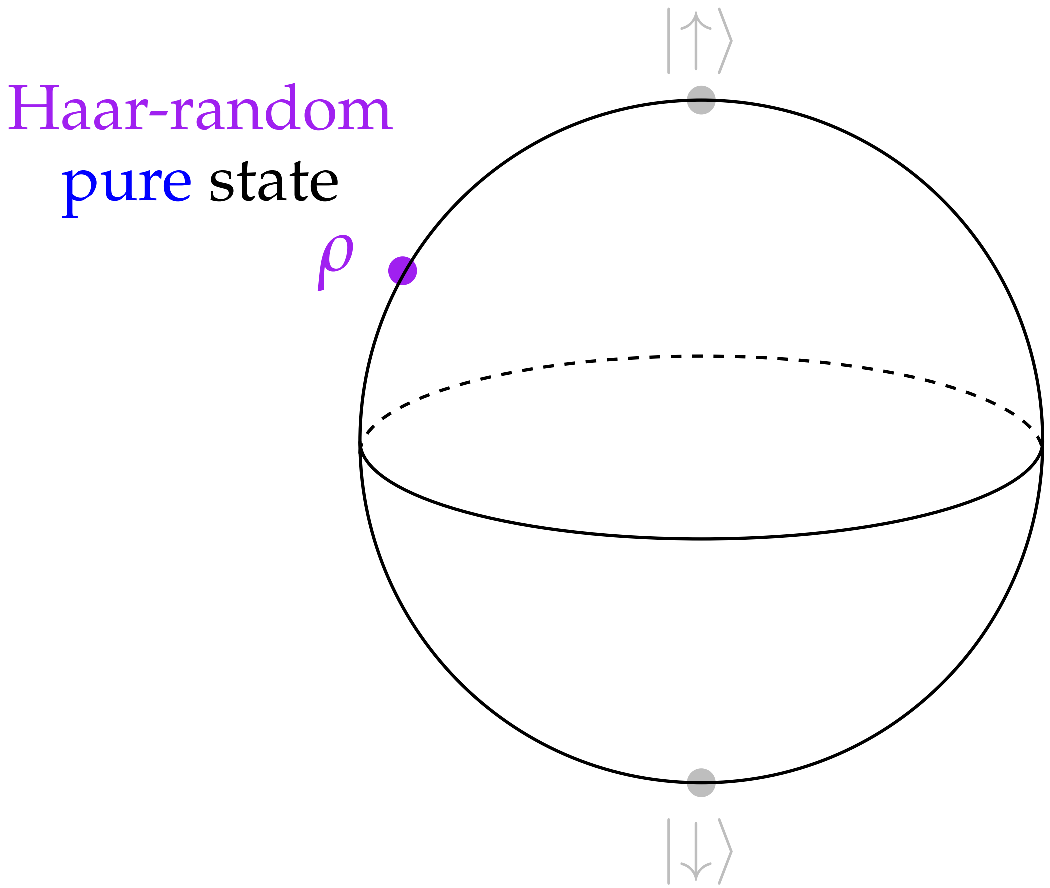
$$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}| \text{ Haar-random pure state}$$

$$\rho_A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ * & & a_d \end{pmatrix} \qquad \rho_B = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ * & & b_d \end{pmatrix} \qquad \rho_C = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ * & & c_d \end{pmatrix}$$

1. Compute joint distribution of marginal diagonals.
2. Recover joint distribution of marginal eigenvalues.

?

Example: Invariant Measures on the Bloch Ball



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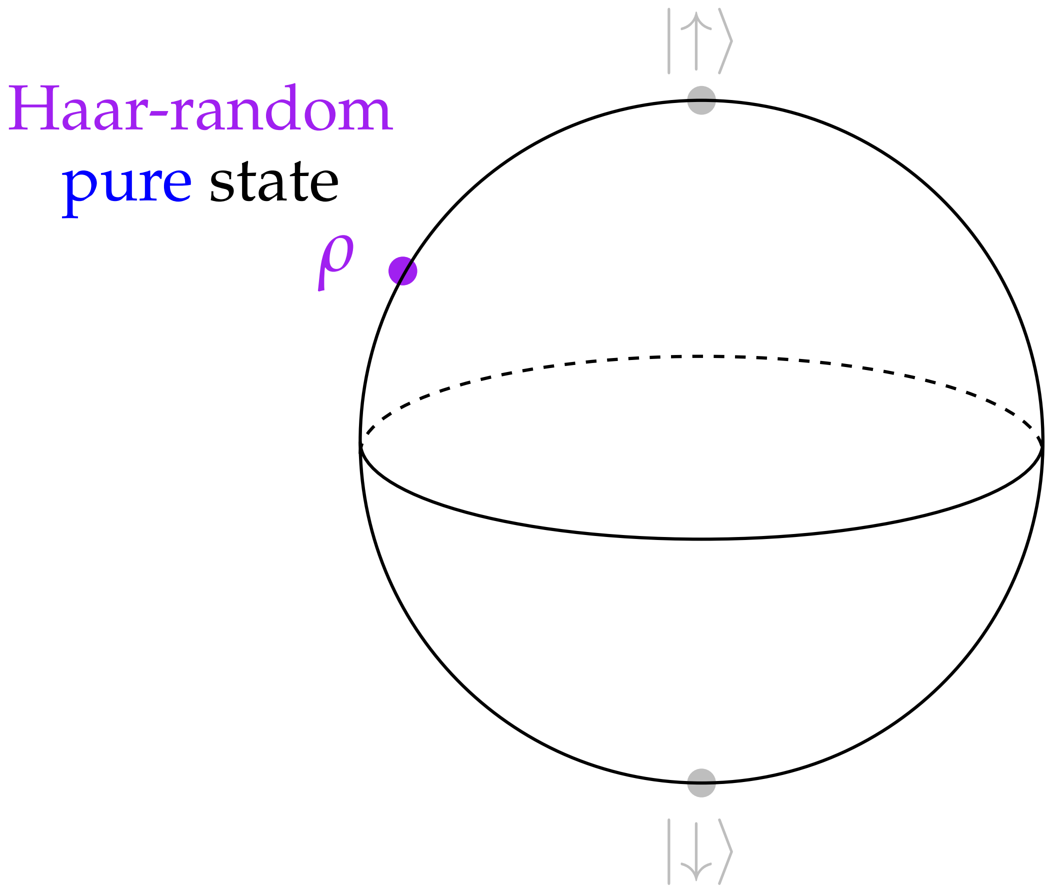

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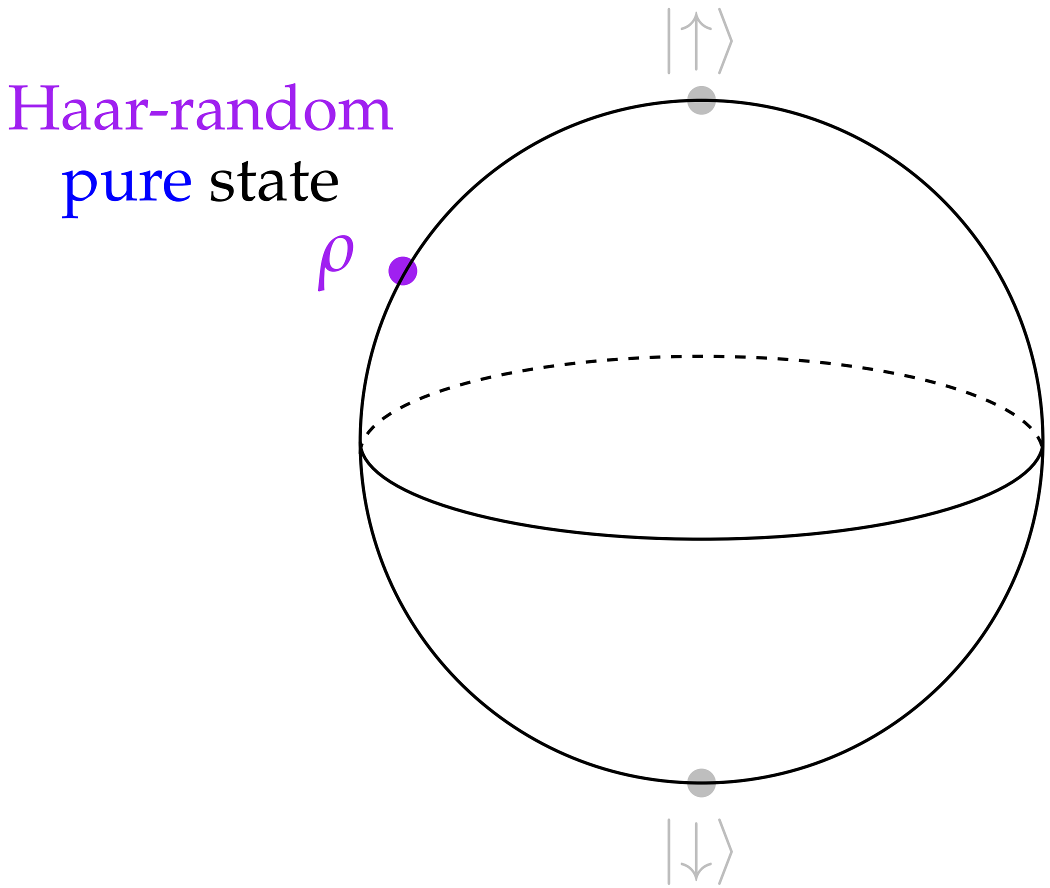
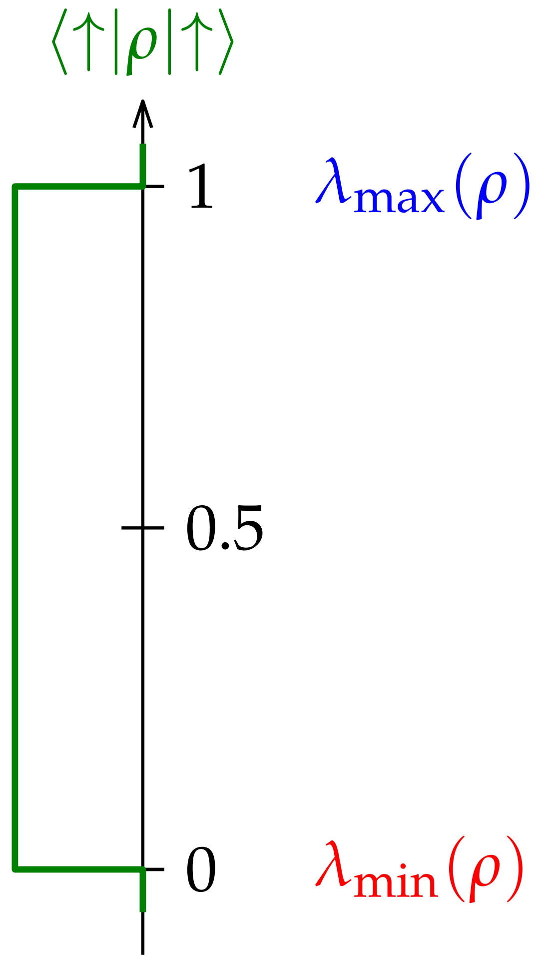
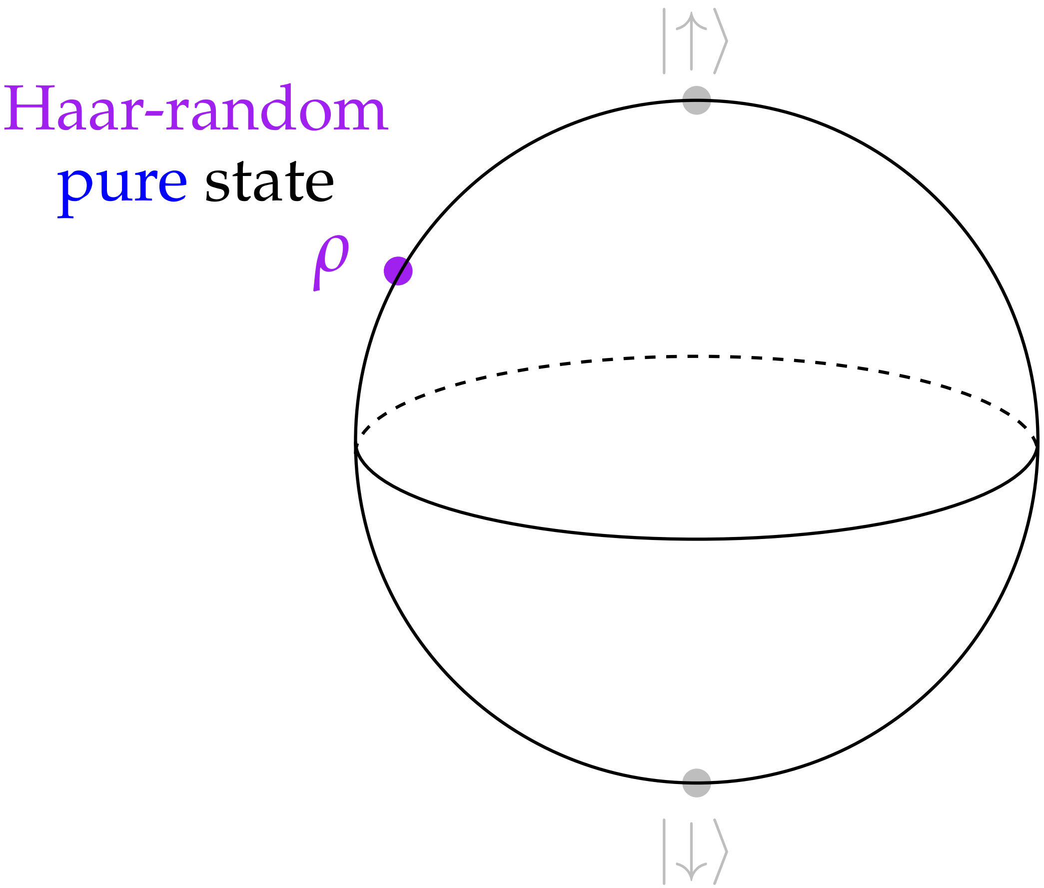
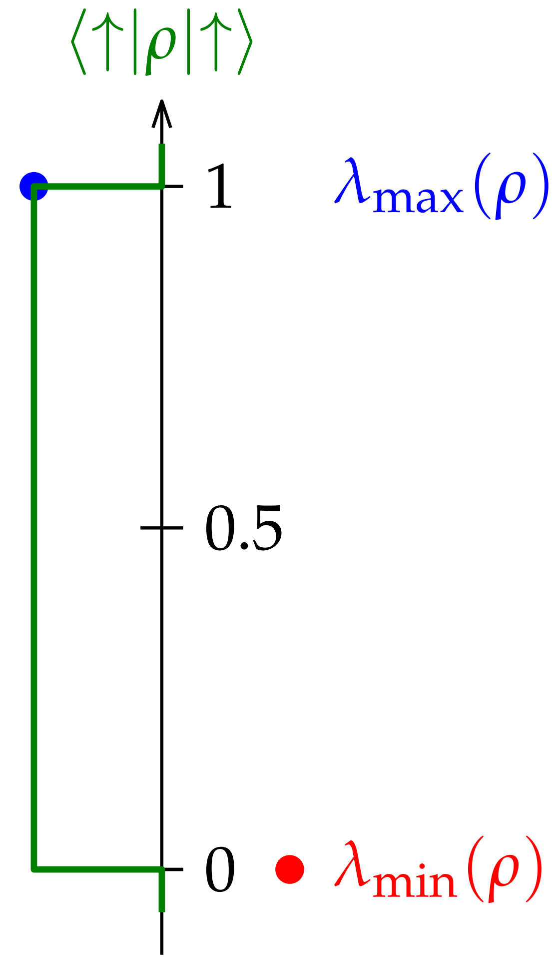




$$\mathbb{P}_{\langle\uparrow|\rho|\uparrow\rangle} = \chi_{[0,1]}$$

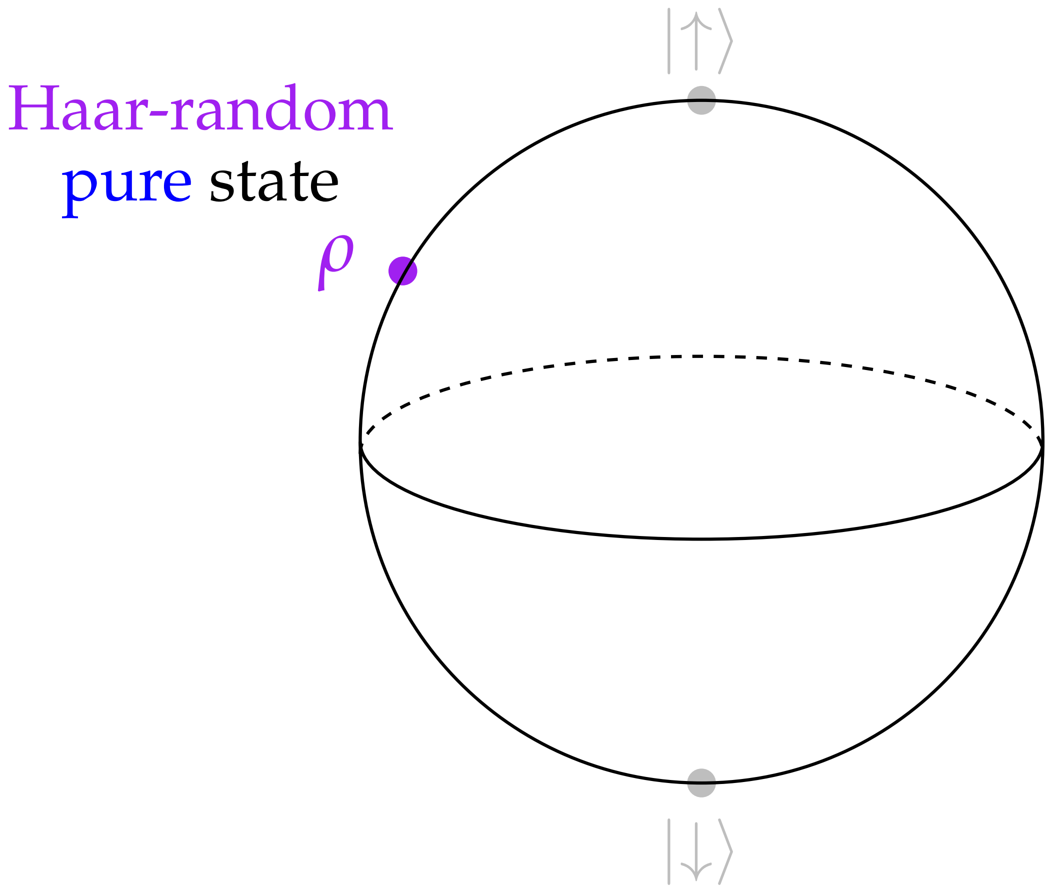
Example: Invariant Measures on the Bloch Ball

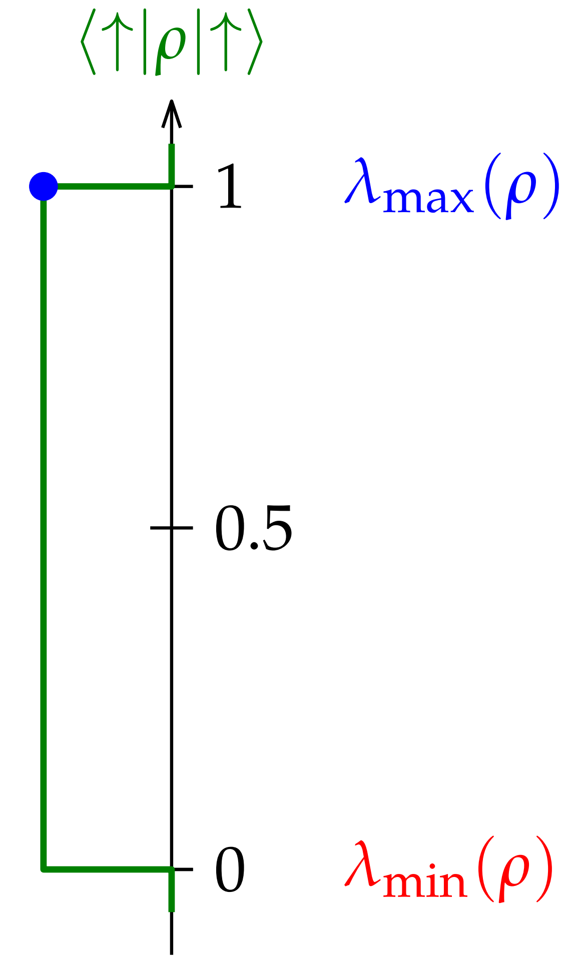




$$-\partial\mathbb{P}_{\langle\uparrow|\rho|\uparrow\rangle} = \delta_1 - \delta_0$$

Example: Invariant Measures on the Bloch Ball

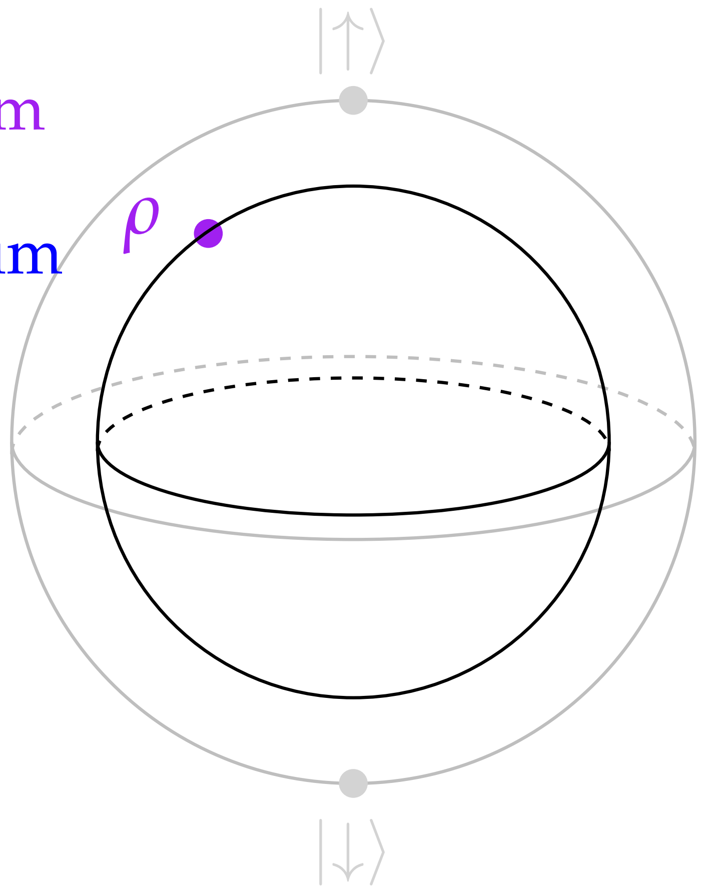


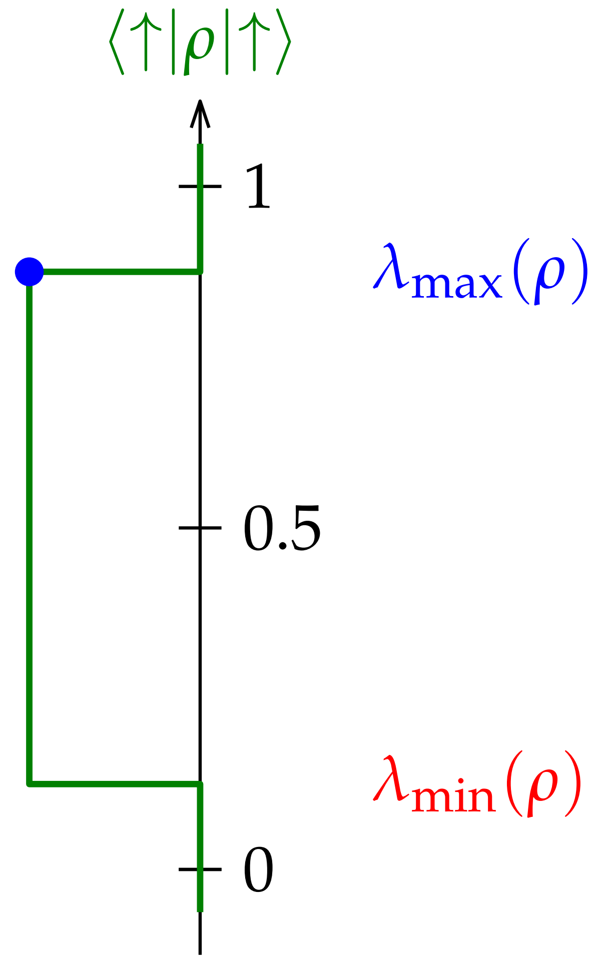


$$-\partial \mathbb{P}_{\langle\uparrow|\rho|\uparrow\rangle}\Big|_{(0.5,\infty)} = \delta_1$$

Example: Invariant Measures on the Bloch Ball

Haar-random state of fixed spectrum

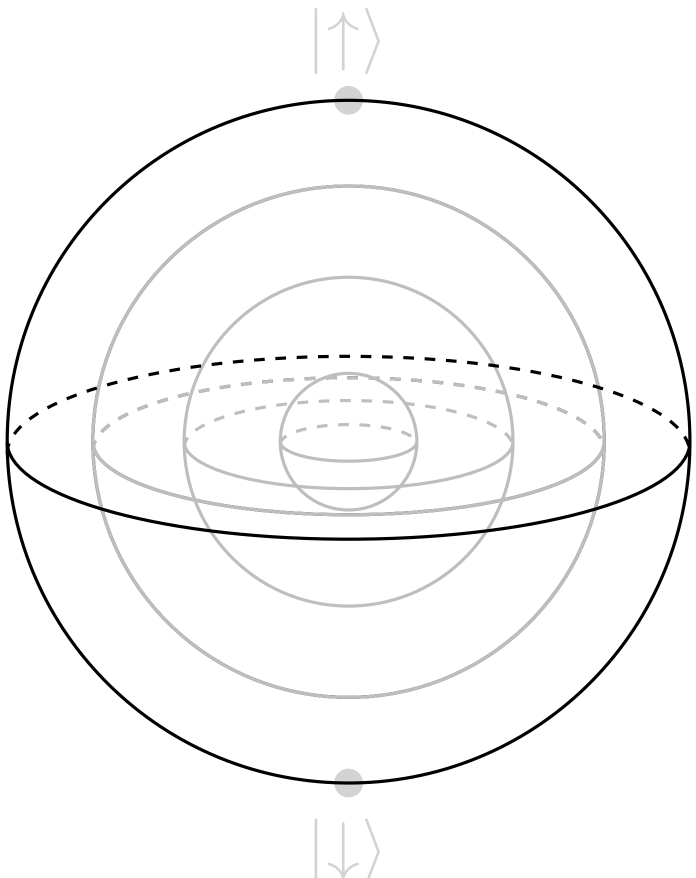


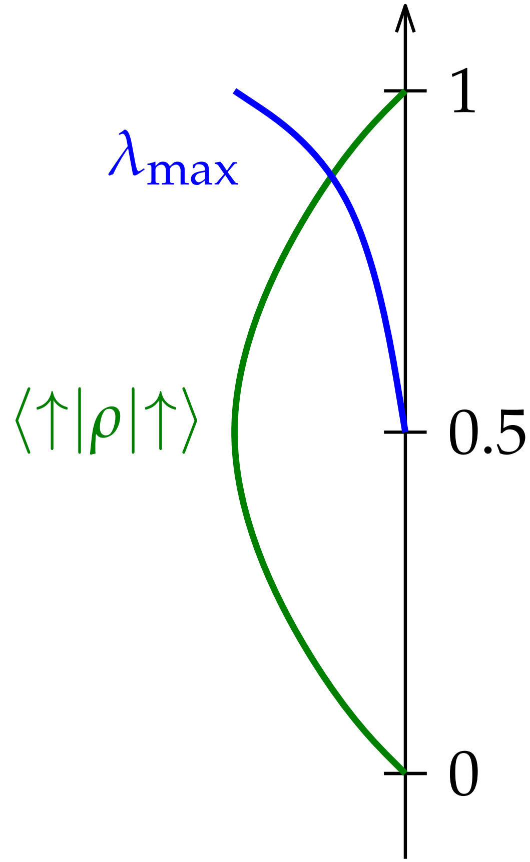


$$-\partial\mathbb{P}_{\langle\uparrow|\rho|\uparrow\rangle}\big|_{(0.5,\infty)} = \frac{1}{\lambda_{\max}-\lambda_{\min}}\delta_{\lambda_{\max}}$$

Example: Invariant Measures on the Bloch Ball

arbitary invariant measure





$$(2\lambda_{\mathrm{max}} - 1)\,(-\partial)\,\mathbb{P}_{\langle\uparrow|\rho|\uparrow\rangle}\big|_{(0.5,\infty)} = \mathbb{P}_{\lambda_{\mathrm{max}}}$$

"derivative principle"

Toy Example: Bosonic Qubits

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$

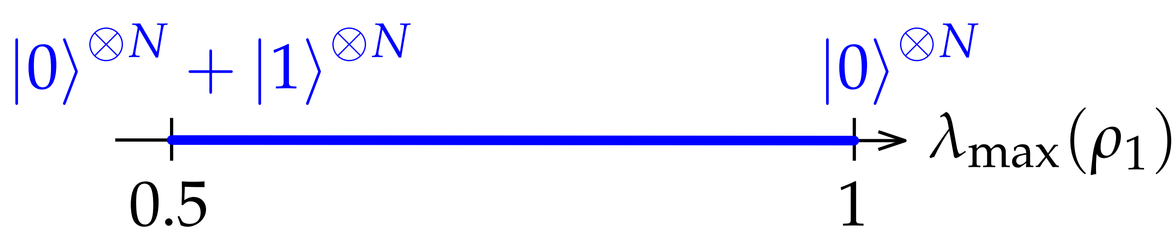
occupation number basis

$\rho = |\psi\rangle\langle\psi|$ random pure state from symmetric subspace

- All one-body marginals equal:

$$\rho_1 = \ldots = \rho_N$$

- Quantum marginal problem is trivial:



Non-trivial part is to determine the probability distribution!

Marginals, Diagonals, and Moments

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$

occupation number basis

$$\rho = |\psi\rangle\langle\psi|$$

↓

ρ_1 quantum marginal → $\langle\uparrow|\rho_1|\uparrow\rangle$ diagonal

↓

$$\lambda_{\max}(\rho_1)$$

Marginals, Diagonals, and Moments

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$

occupation number basis

$\rho = |\psi\rangle\langle\psi|$ $\longrightarrow$ $p_n = \langle n|\rho|n\rangle$ diagonal

$\downarrow$ $\downarrow$

ρ_1 quantum marginal $\longrightarrow$ $\langle\uparrow|\rho_1|\uparrow\rangle$ diagonal

$$= \frac{1}{N}\sum_{n=0}^{N} p_n\, n$$

$\downarrow$

$\lambda_{\max}(\rho_1)$

Marginals, Diagonals, and Moments

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$ occupation number basis

$$\rho = |\psi\rangle\langle\psi| \quad \longrightarrow \quad p_n = \langle n|\rho|n\rangle \text{ diagonal}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\rho_1 \text{ quantum marginal} \quad \longrightarrow \quad \langle\uparrow|\rho_1|\uparrow\rangle \text{ diagonal}$$

$$= \frac{1}{N}\sum_{n=0}^{N} p_n\, n$$

$$\langle\uparrow|\rho_1|\uparrow\rangle \equiv \text{first moment of one-body observable}$$

$$|\uparrow\rangle\langle\uparrow| \otimes \mathbb{I} + \ldots + \mathbb{I} \otimes |\uparrow\rangle\langle\uparrow|$$

Marginals, Diagonals, and Moments

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$

occupation number basis

$\rho = |\psi\rangle\langle\psi|$ $\longrightarrow$ $p_n = \langle n|\rho|n\rangle$ diagonal

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ρ_1 quantum marginal $\longrightarrow$ $\langle\uparrow|\rho_1|\uparrow\rangle$ diagonal

$$= \frac{1}{N}\sum_{n=0}^{N} p_n\, n$$

$\rho_1 \equiv$ first moments of all one-body observables

Computing the Diagonal Distribution

$$\mathrm{Sym}^N(\mathbb{C}^2) = \bigoplus_{n=0}^{N} \mathbb{C}|n\rangle$$

$\rho = |\psi\rangle\langle\psi|$ random pure state

$\downarrow$

$(p_n) = (\langle n|\rho|n\rangle)$ Lebesgue-random in standard simplex

$$\Delta_N = \left\{ (p_0, \ldots, p_N) : \sum_{n=0}^{N} p_n = 1, p_n \geq 0 \right\}$$

Sampling from Δ_N:

- $X_1, \ldots, X_N \sim [0,1]$ uniform, independent, $X_0 := 1$, $X_{N+1} := 0$
- $X_{\pi(1)} > \ldots > X_{\pi(N)}$ sorted
- $p_k := X_{\pi(k)} - X_{\pi(k+1)}$ has correct distribution

Computing the Marginal Diagonal Distribution

Sampling from Δ_N:

- $X_1, \ldots, X_N \sim [0,1]$ uniform, independent, $X_0 := 1$, $X_{N+1} := 0$
- $X_{\pi(1)} > \ldots > X_{\pi(N)}$ sorted
- $p_k := X_{\pi(k)} - X_{\pi(k+1)}$ has correct distribution

$$\langle\uparrow|\rho_1|\uparrow\rangle = \frac{1}{N}\sum_{n=0}^{N} p_n\, n$$

$$= \frac{1}{N}\left(X_{\pi(1)} - X_{\pi(2)} + 2X_{\pi(2)} - 2X_{\pi(3)} + \ldots\right)$$

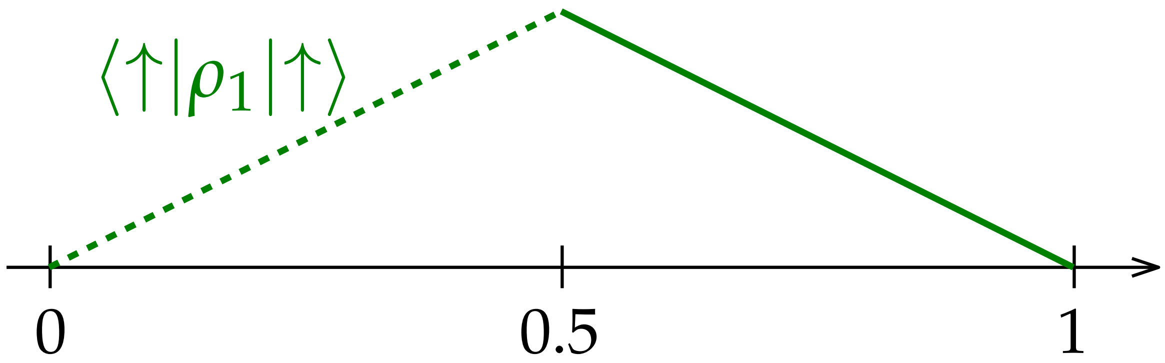
$$= \frac{1}{N}\left(X_1 + \ldots + X_N\right)$$

Sum of N i.i.d. random variables!

Recovering the Marginal Eigenvalue Distribution

$$(2\lambda_{\max} - 1)\,(-\partial)\,\mathbb{P}_{\langle\uparrow|\rho_1|\uparrow\rangle}\Big|_{(0.5,\infty)} = \mathbb{P}_{\lambda_{\max}}$$

Example ($N = 2$):



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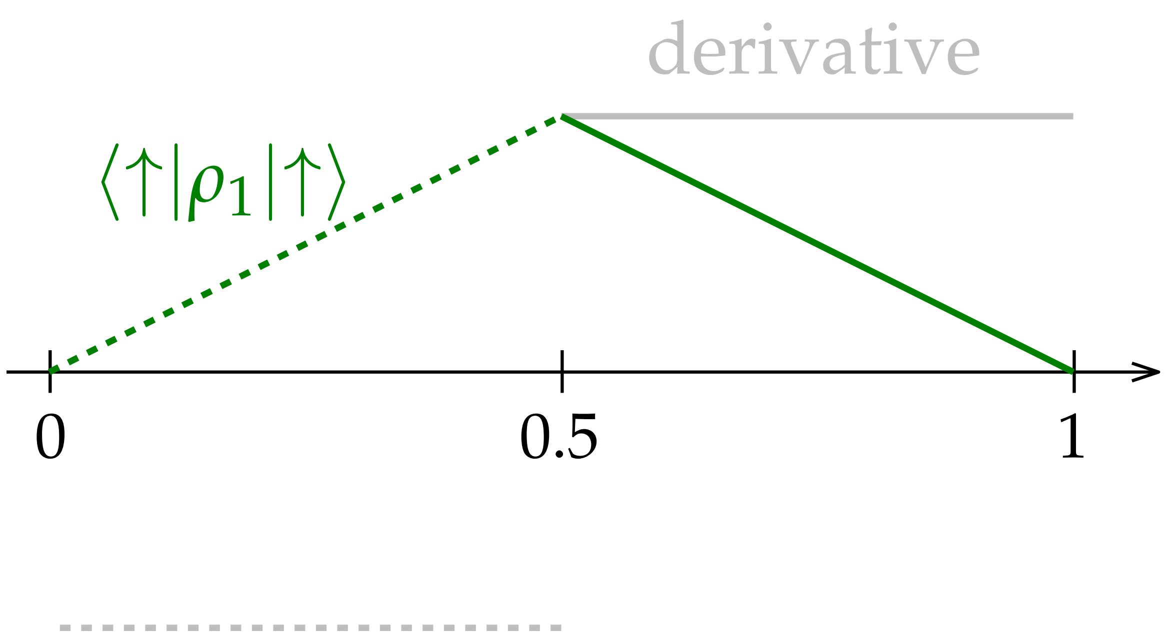

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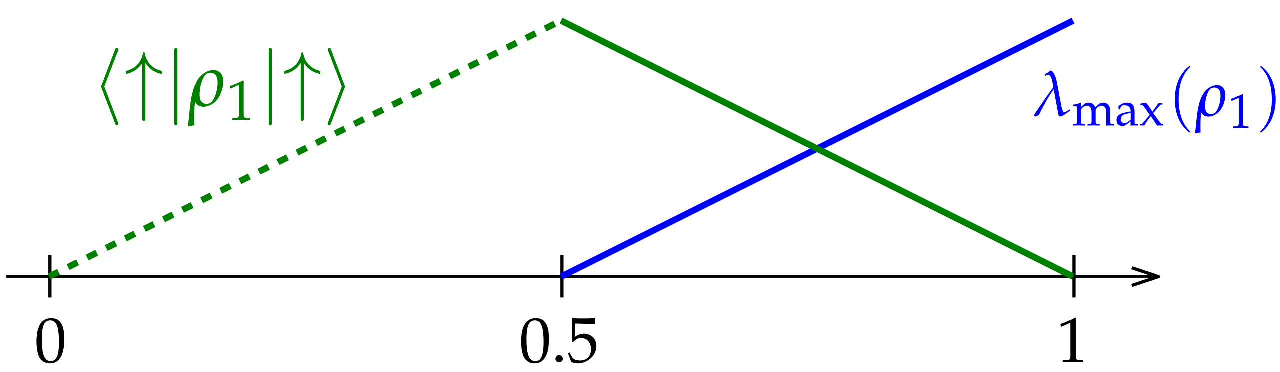


The General Algorithm

1. Compute joint distribution of marginal diagonals:

$$\rho_A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ * & & a_d \end{pmatrix} \qquad \rho_B = \begin{pmatrix} b_1 & & * \\ & \ddots & \\ * & & b_d \end{pmatrix} \qquad \rho_C = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ * & & c_d \end{pmatrix}$$

pushforward of Lebesgue measure on standard simplex along linear map $\longrightarrow$ piecewise polynomial density [Boysal–Vergne]

2. Recover joint distribution of marginal eigenvalues:

$$\mathbb{P}_{\mathrm{eig}} = \mathrm{poly} \cdot \left(\prod_{\alpha>0} -\partial_\alpha \right) \mathbb{P}_{\mathrm{diag}} \Big|_{\mathfrak{t}^*_{>0}}$$

normalization factor

positive roots

positive Weyl chamber

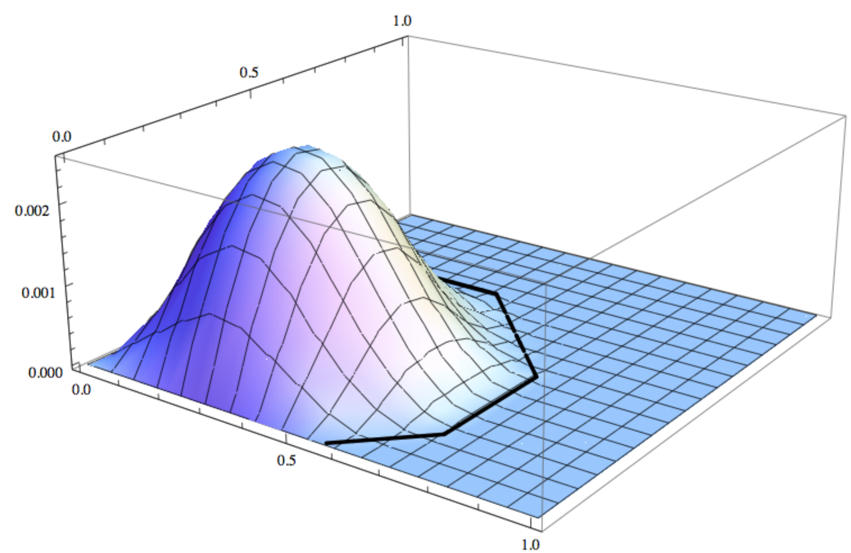
[Harish-Chandra]

More Examples

- $\mathrm{Sym}^N(\mathbb{C}^2)$:

$$\frac{(\lambda_{\max} - 0.5)_+}{(N-2)!N!} \sum_{k=0}^{N} (-1)^{k+1} \binom{N}{k} (N\lambda_{\max} - k)_+^{N-2}$$

- $\mathbb{C}^2 \otimes \mathbb{C}^2$ with global spectrum $\vec{\lambda}_{AB} = (\frac{4}{7}, \frac{2}{7}, \frac{1}{7}, 0)$:



3. The Branching Problem

The Branching Problem

$H \subseteq G$ compact, connected Lie groups

$V_{G,\lambda}$, $V_{H,\mu}$ irreducible representations

$$V_{G,\lambda}\big|_H = \bigoplus_\mu m_\mu^\lambda \, V_{H,\mu}$$

How to compute branching multiplicities efficiently?

The Branching Problem

$H \subseteq G$ compact, connected Lie groups

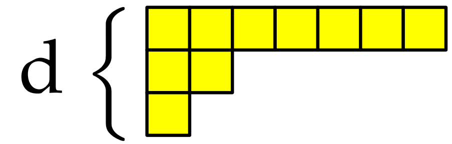
$V_{G,\lambda}$, $V_{H,\mu}$ irreducible representations

$$V_{G,\lambda}\big|_H = \bigoplus_\mu m_\mu^\lambda \, V_{H,\mu}$$

How to compute branching multiplicities efficiently?

Result: Poly-time algorithm for any fixed $H \subseteq G$.

Given highest weights λ, μ encoded as bitstrings, the algorithm computes the multiplicity m_μ^λ in polynomial time.



Why care?

Poly-time algorithm for fixed d

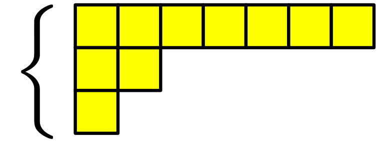
Cochet (2005)

Kostka numbers
$T(d) \subseteq U(d)$

Cochet (2005), De Loera & McAllister (2006)

Littlewood–Richardson coefficients
$U(d) \subseteq U(d) \times U(d)$

Kronecker coefficients
$U(d) \times U(d) \subseteq U(d^2)$

d 

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Christandl, Doran & W. (2012)

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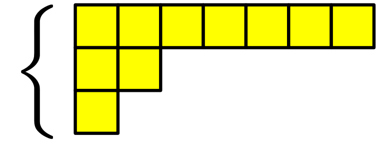
unified by the result

d 

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Poly-time algorithm for fixed d

$\#P$-hardness for variable d

Poly-time algorithm for fixed d		$\#P$-hardness for variable d
Cochet (2005)	**Kostka numbers** $T(d) \subseteq U(d)$	*Narayanan (2006)*
Cochet (2005), De Loera & McAllister (2006)	**Littlewood–Richardson coefficients** $U(d) \subseteq U(d) \times U(d)$	*Narayanan (2006)*
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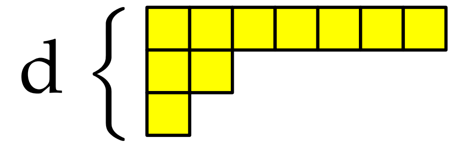
unified by the result

Why care?



Poly-time algorithm for fixed d		**#P-hardness for variable d**
Cochet (2005)	**Kostka numbers** $T(d) \subseteq U(d)$	*Narayanan (2006)*
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Christandl, Doran & W. (2012)	**Kronecker coefficients** $U(d) \times U(d) \subseteq U(d^2)$	*Bürgisser & Ikenmeyer (2008)*

Conjecture (Geometric Complexity Theory):
Positivity can be decided in poly-time.

Branching Problem for Tori

$$G = U(1)^r$$

all compact, connected Abelian Lie groups

All irreducible representations are **one-dimensional** and of the form

$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix} \cdot |1\rangle = z_1^{k_1} \cdots z_r^{k_r} |1\rangle$$

Labeled by their weight $\omega = (k_1, \ldots, k_r) \in \mathbb{Z}^r$.

Branching Problem for Tori

$$G = U(1)^r$$

$$H = U(1)^s$$

(Thus) any homomorphism $H \to G$ is of the form

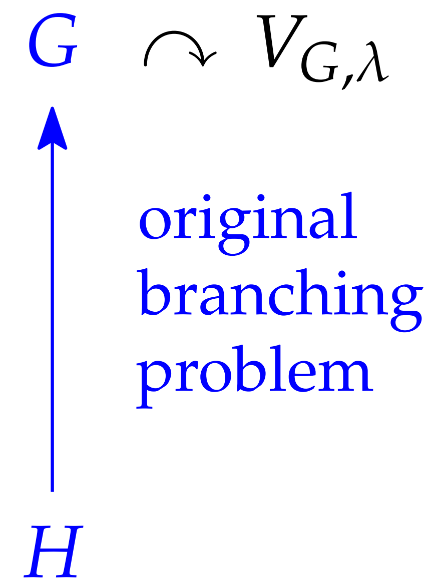
$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_s \end{pmatrix} \mapsto \begin{pmatrix} z_1^{k_{1,1}} \cdots z_s^{k_{s,1}} & & \\ & \ddots & \\ & & z_1^{k_{1,r}} \cdots z_s^{k_{s,r}} \end{pmatrix}$$

for an integer matrix $\Omega = (k_{i,j}) \in \mathbb{Z}^{s \times r}$.

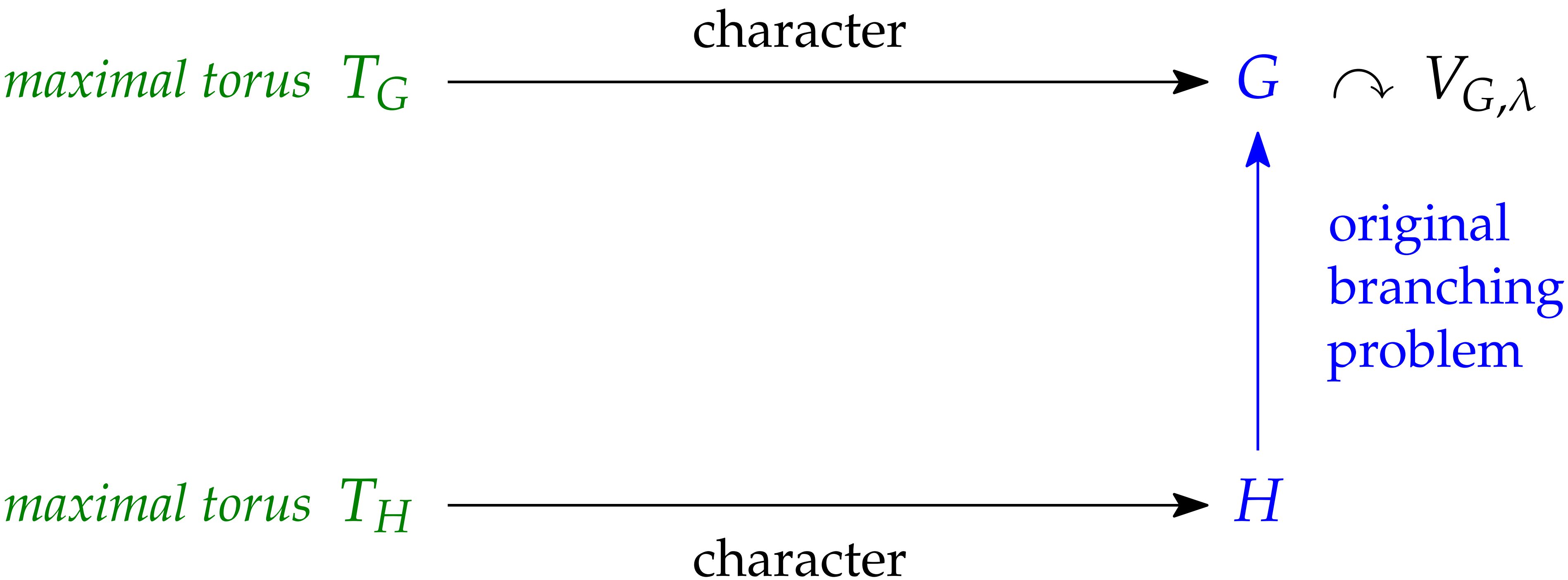
$$\boxed{V_{G,\omega}\big|_H = V_{H,\Omega\omega}}$$

linear map!

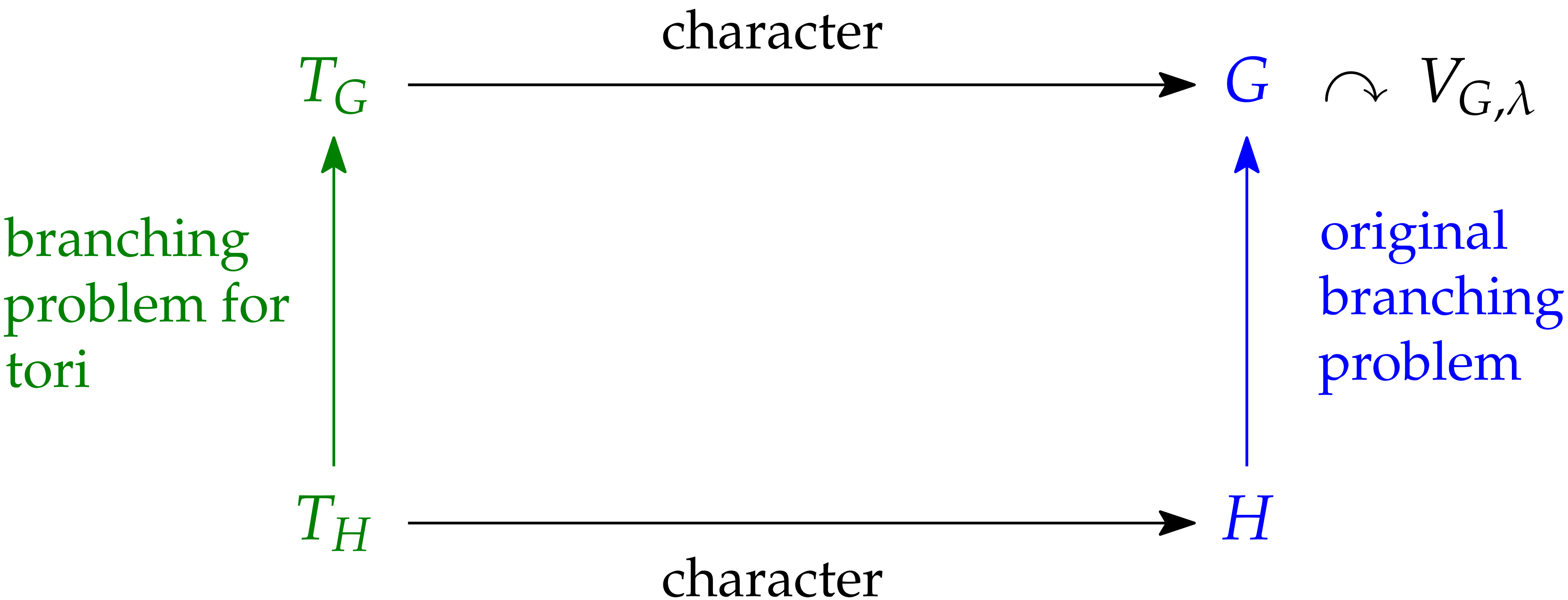
Strategy



Strategy


character
maximal torus T_G
G
$\curvearrowright V_{G,\lambda}$
original
branching
problem
maximal torus T_H
H
character

Strategy


character
T_G
G
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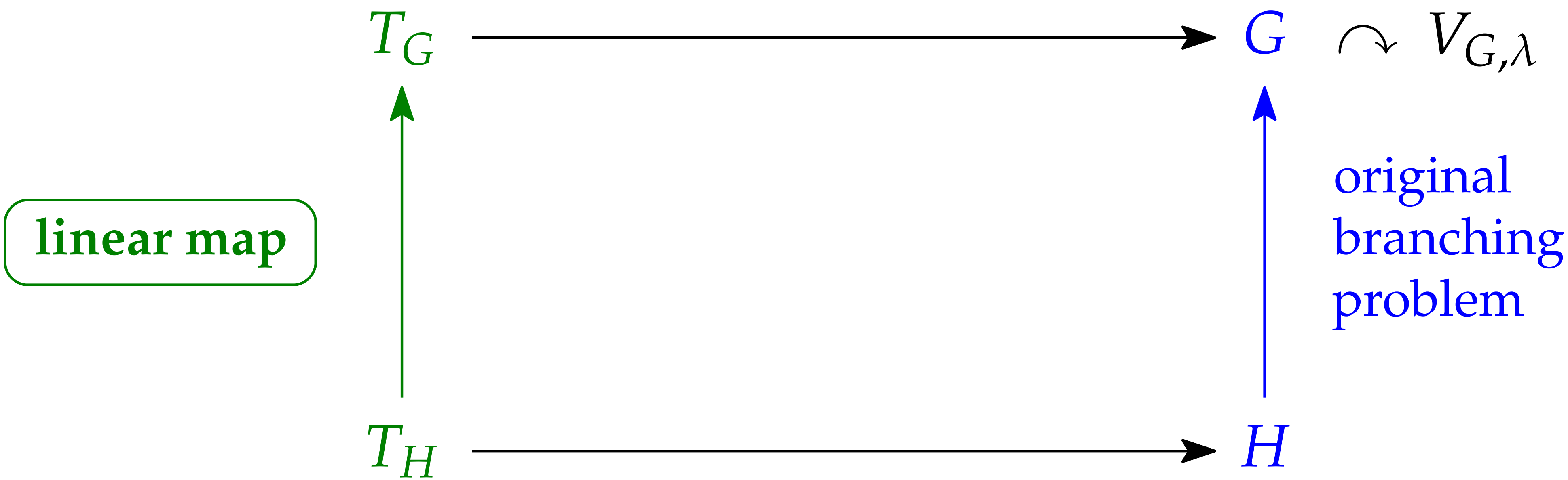
branching problem for tori
original branching problem
T_H
H
character

Strategy

multiplicity of weight = # points in convex polytope $\Delta(\omega, \lambda)$

Kostant (1959)
Billey et al (2004)
Bliem (2008)



Strategy

Barvinok's algorithm!

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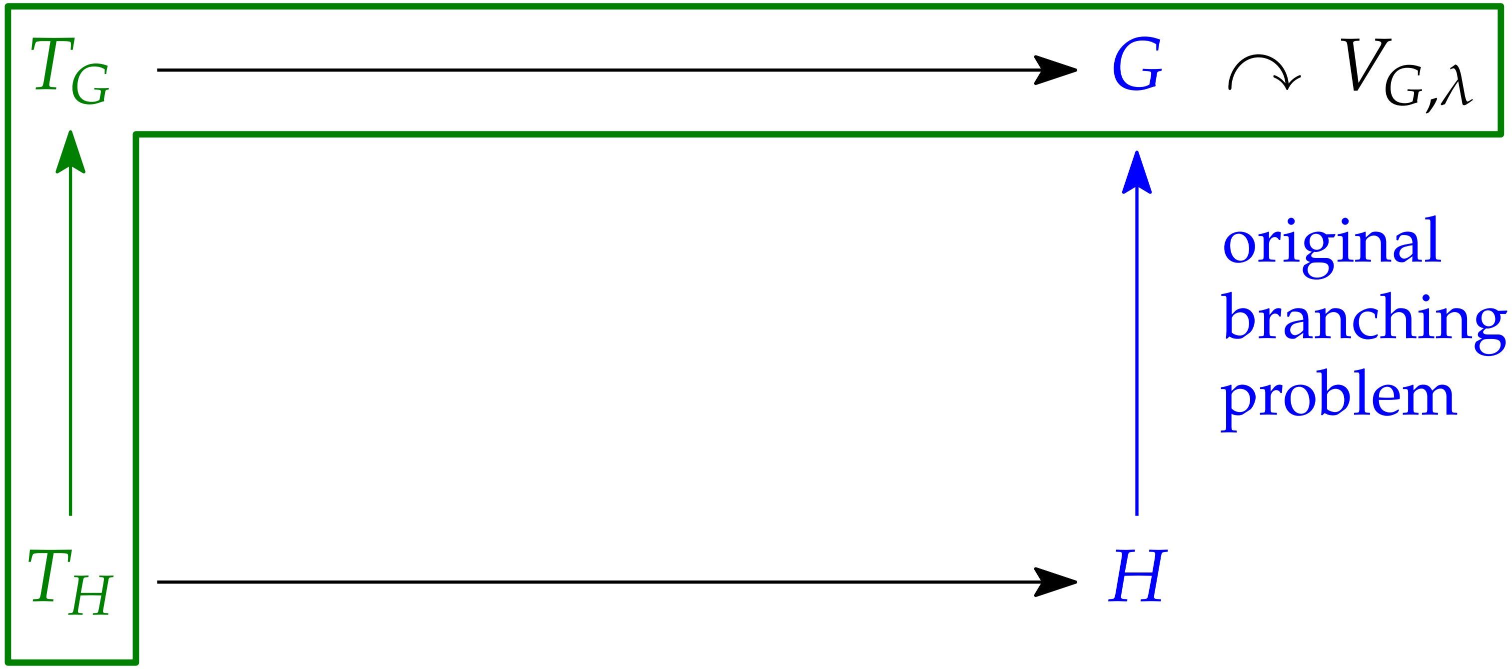
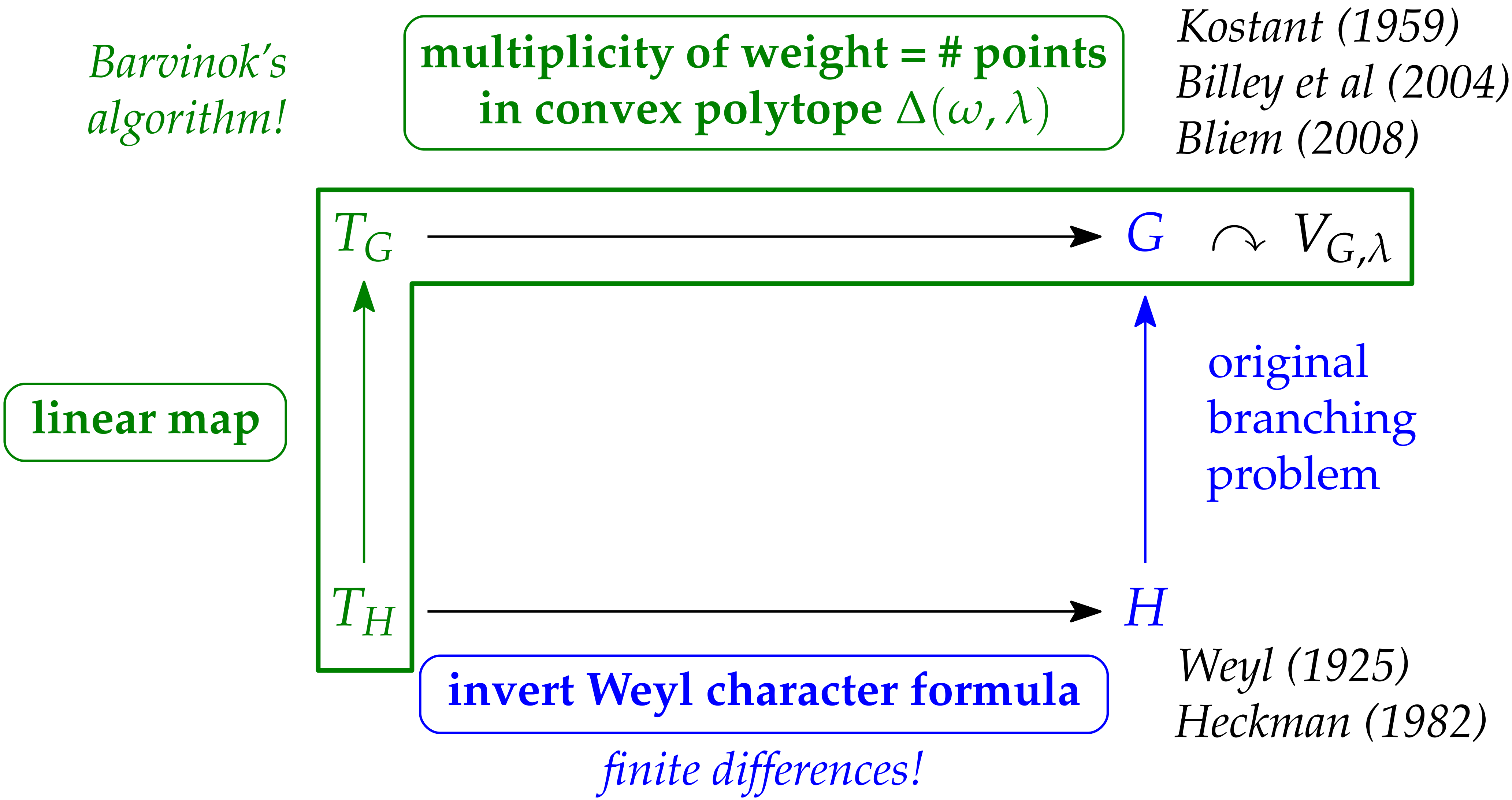
linear map



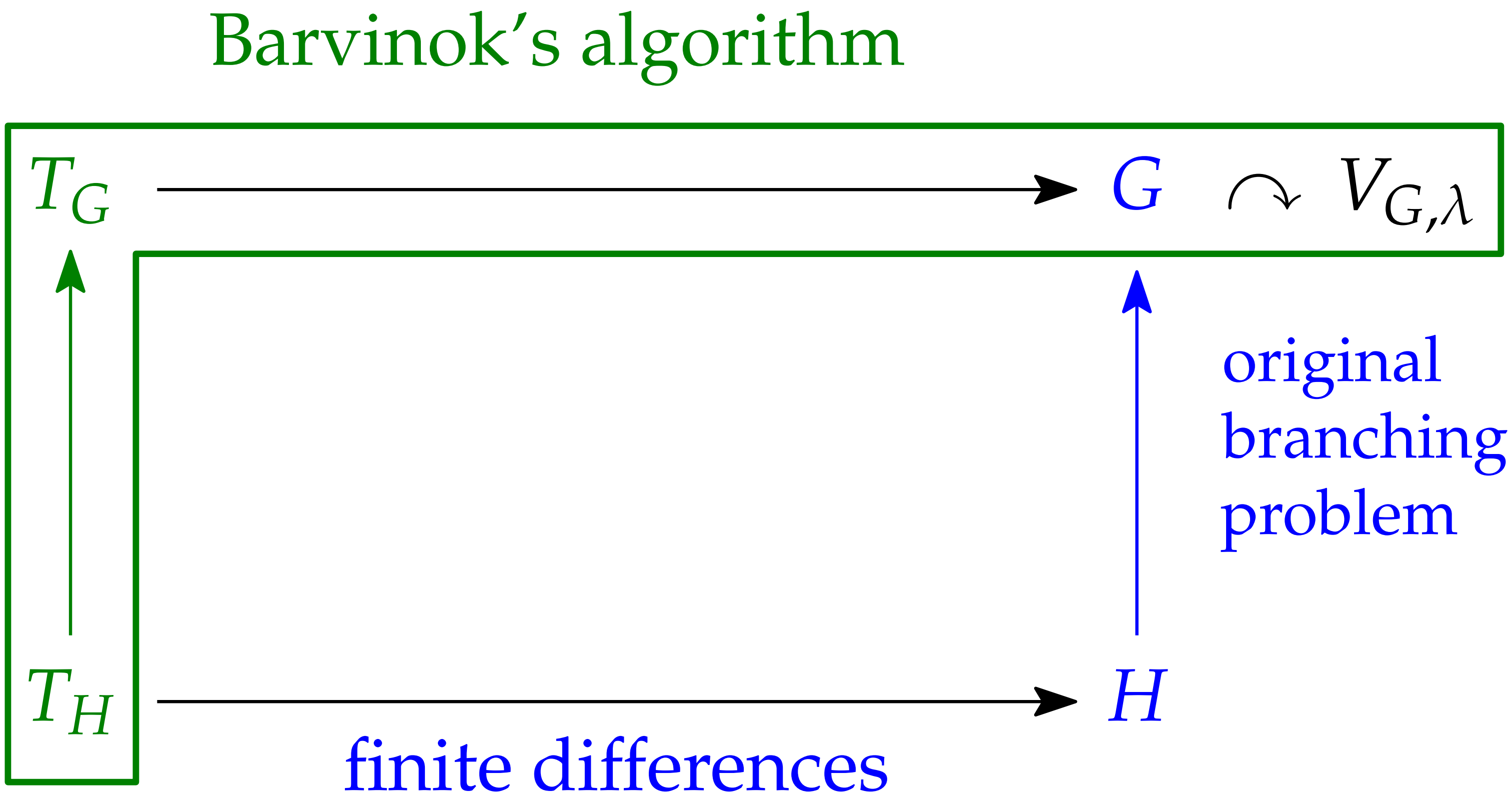
original branching problem

Strategy



Barvinok's algorithm!
multiplicity of weight = # points in convex polytope $\Delta(\omega, \lambda)$
Kostant (1959)
Billey et al (2004)
Bliem (2008)
T_G
$G \curvearrowright V_{G,\lambda}$
linear map
original branching problem
T_H
H
invert Weyl character formula
Weyl (1925)
Heckman (1982)
finite differences!

Strategy



Barvinok's algorithm
T_G
G
$\curvearrowright V_{G,\lambda}$
original
branching
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H
finite differences

Example: Quantizing the Bloch Sphere

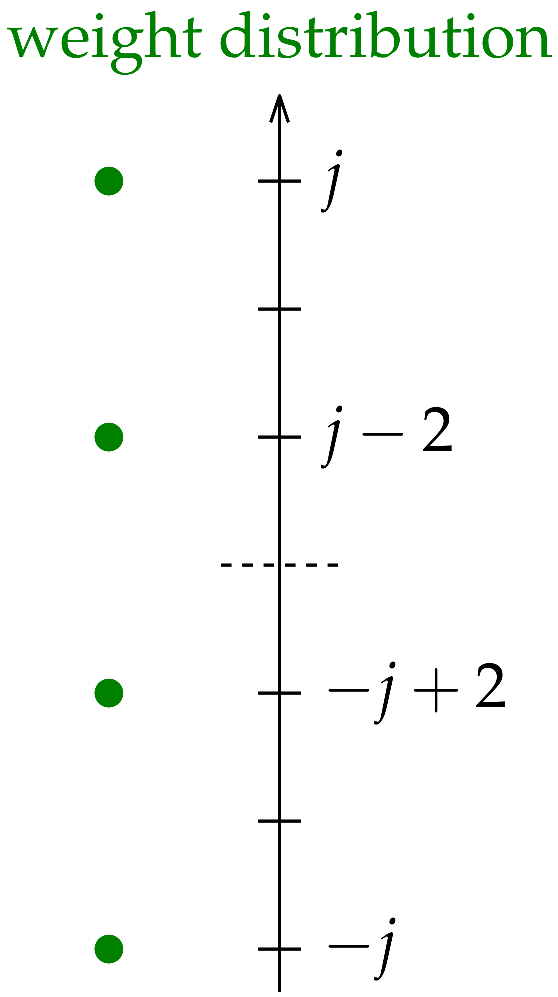
The irreducible representations of $\mathrm{SU}(2)$ are

$$V_j = \mathrm{Sym}^j(\mathbb{C}^2)$$

labeled by their spin $j = 0, 1, \ldots$

Maximal torus $\{\begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}\} \cong U(1)$, irreducible representations are labeled by weight $k \in \mathbb{Z}$.

eigenvalues of $\sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, occupation numbers



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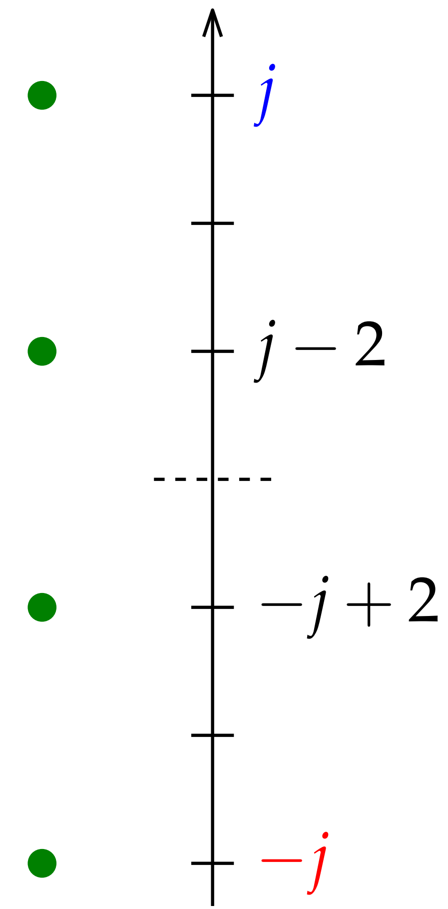
weight distribution



$$\mu_{\text{weight}} = \delta_{-j} + \ldots + \delta_j$$

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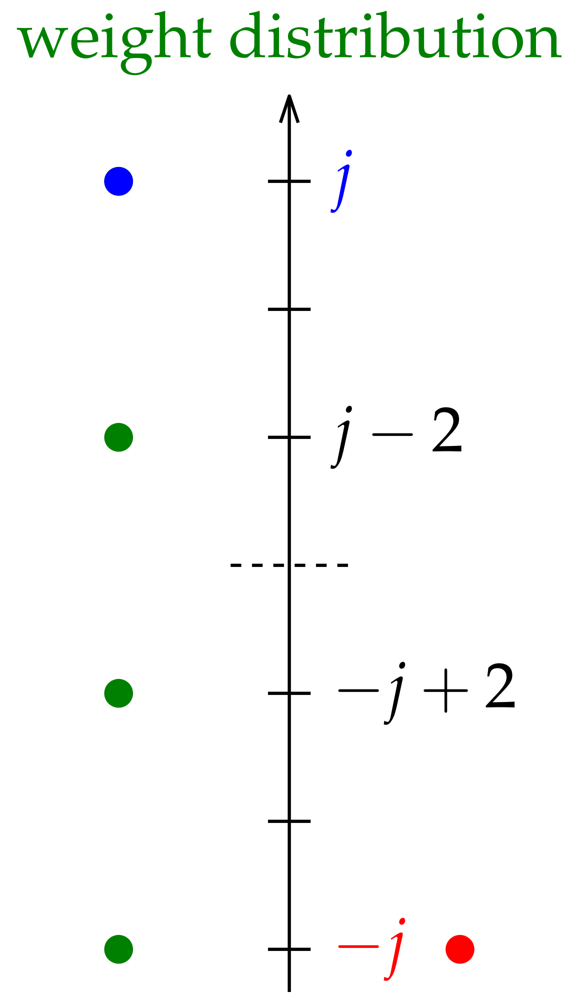

$$-\Delta\mu_{\text{weight}} = \delta_j - \delta_{-j}$$

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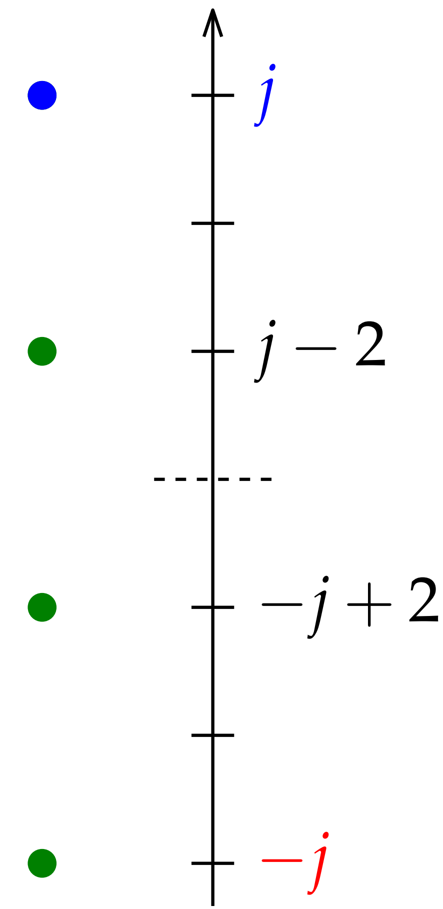

$$-\Delta \mu_{\text{weight}}\big|_{[0,\infty)} = \delta_j$$

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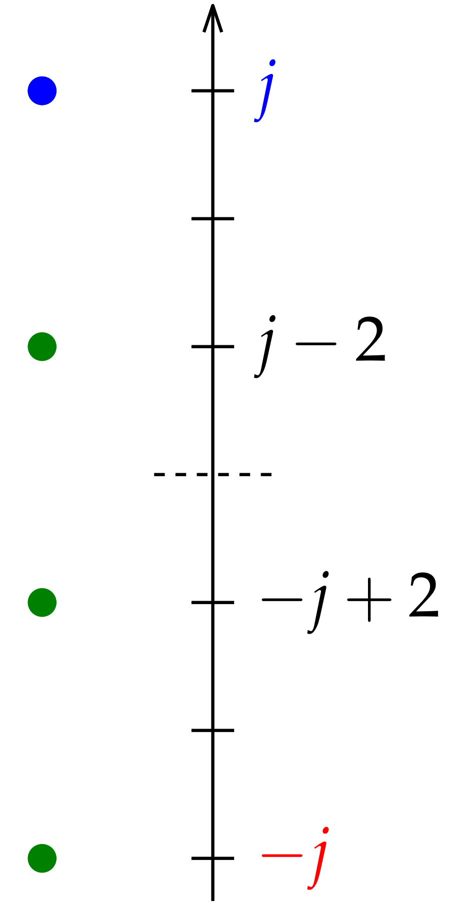

$$-\Delta\mu_{\text{weight}}\Big|_{[0,\infty)} = \mu_{\text{spin}}$$

“finite difference formula”

The General Algorithm

$$V_{G,\lambda}\big|_H = \bigoplus_\mu m_\mu^\lambda \, V_{H,\mu}$$

1. Compute weight multiplicities for $T_H \curvearrowright V_{G,\lambda}$

   # points in convex polytope $\Delta(\omega,\lambda) \longrightarrow$ Barvinok's algorithm

[Kostant, Bliem]

2. Recover branching coefficients:

$$\mu_{\text{branching}} = \left(\prod_{\alpha>0} -\Delta_\alpha \right) \mu_{\text{weights}} \Big|_{\mathfrak{t}^*_{>0}}$$

positive roots

positive Weyl chamber

[Weyl, Heckman]

$\rightarrow$ poly-time algorithm

4. The Semiclassical Limit

The Semiclassical Limit

Branching Problem

$K = U(d) \times U(d) \times U(d)$
$\subseteq G = U(d^3) = U(\mathcal{H})$

$V_{G,\lambda} = V_{G,[k]} = \mathrm{Sym}^k(\mathcal{H})$

Random Marginal Problem

$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$

$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}|$ random

The Semiclassical Limit

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$$\subseteq G = U(d^3) = U(\mathcal{H})$$

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T_G-weight multiplicities:
occupation numbers,
integral points in $k\Delta_{d^3-1}$

scaling limit →

Random Marginal Problem

$$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$$

$$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}| \text{ random}$$

diagonal distribution:
Lebesgue measure on Δ_{d^3-1}

The Semiclassical Limit

Branching Problem		**Random Marginal Problem**
$K = U(d) \times U(d) \times U(d) \subseteq G = U(d^3) = U(\mathcal{H})$		$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$
$V_{G,\lambda} = V_{G,[k]} = \mathrm{Sym}^k(\mathcal{H})$		$\rho_{ABC} = \lvert\psi_{ABC}\rangle\langle\psi_{ABC}\rvert$ random
T_G-weight multiplicities: occupation numbers, integral points in $k\Delta_{d^3-1}$	scaling limit →	diagonal distribution: Lebesgue measure on Δ_{d^3-1}
linear ↓ map		linear ↓ map
T_H-weight multiplicities	→	marginal diagonal distribution

The Semiclassical Limit

Branching Problem

$$K = U(d) \times U(d) \times U(d)$$
$$\subseteq G = U(d^3) = U(\mathcal{H})$$

$$V_{G,\lambda} = V_{G,[k]} = \mathrm{Sym}^k(\mathcal{H})$$

T_G-weight multiplicities: occupation numbers, integral points in $k\Delta_{d^3-1}$

linear ↓ map

T_H-weight multiplicities

finite ↓ differences

branching coefficients ("Kronecker coefficients")

Random Marginal Problem

$$\mathcal{H} = \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$$

$$\rho_{ABC} = |\psi_{ABC}\rangle\langle\psi_{ABC}| \text{ random}$$

scaling limit →

diagonal distribution: Lebesgue measure on Δ_{d^3-1}

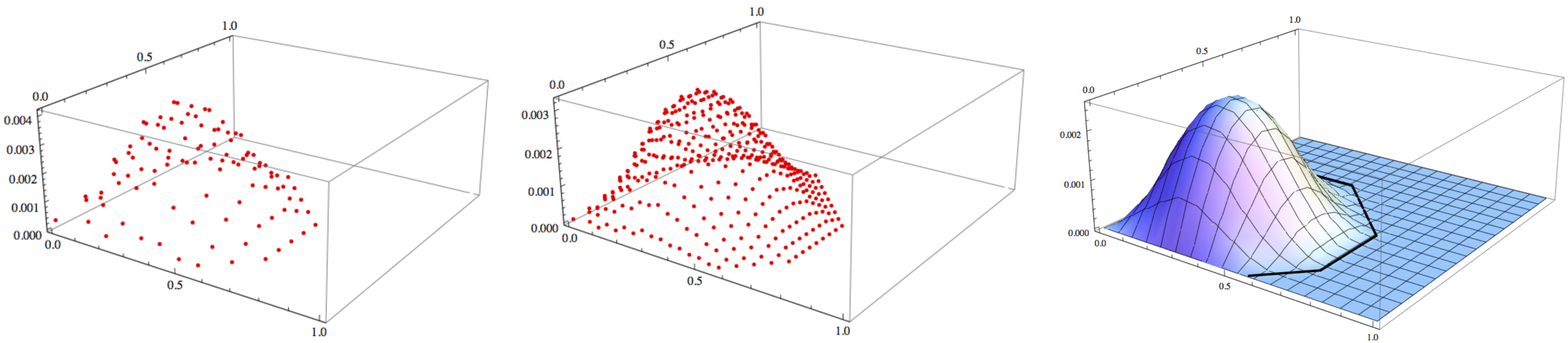
linear ↓ map

marginal diagonal distribution

partial ↓ differences

marginal eigenvalue distribution

Thanks for Your Attention!



Quantum Marginals $\leftrightarrow$ Classical Marginals

Branching Multiplicities $\leftrightarrow$ Weight Multiplicities

Asymptotics of random marginals? Onset of asymptotics?
Positivity of branching multiplicities?

arXiv:1204.0741 and FOCS 2012