An introduction to discrete phase space and Schur-Weyl duality for the Clifford group

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Plan for today

1. Introduction to discrete phase space
   *Pauli & Clifford group, stabilizer states, motivation*

2. Schur-Weyl duality for the Clifford group
   *higher moments, property testing, de Finetti, …*
Quantum optics motivation

Linear quantum optics described by Gaussian unitaries $U_G$ (beam splitters, squeezing...), generate Gaussian states $|\psi\rangle = U_G |0\rangle$ (coherent, squeezed states...)

Phase space for $n$ optical modes: $\mathbb{R}^{2n} \ni \mathbf{v} = (q, p)$

- displacement operators $D_v = e^{i(pQ - qP)}$
- commutation relations: $D_v D_w = e^{i[v, w]} D_w D_v \propto D_{v+w}$
- Gaussian unitaries act by symplectic transformations: $U_G D_v U_G^\dagger \propto D_{\Gamma v}$

Phase space distributions:

- characteristic fn. $\chi_\rho (\mathbf{v}) = \text{tr}[\rho D_v]$ and Wigner function
- Gaussian for Gaussian states $\sim$ mean & covariance

Highly useful – let’s find a similar formalism in finite dimensions!
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*Highly useful – let’s find a similar formalism in finite dimensions!*)
Pauli operators and discrete phase space

Discrete phase space for \( n \) qubits: \( \mathbb{F}_2^{2n} \ni \mathbf{v} = (q, p) \).

Pauli operators:

\[
P_{\mathbf{v}} = P_{v_1} \otimes \ldots \otimes P_{v_n} \text{ where } P_{00} = I, P_{01} = X, P_{10} = Z, P_{11} = Y
\]

- commutation relations: \( P_{\mathbf{v}} P_{\mathbf{w}} = (-1)^{[\mathbf{v}, \mathbf{w}]} P_{\mathbf{w}} P_{\mathbf{v}} \propto P_{\mathbf{v}+\mathbf{w}} \pmod{2} \)
- generate Pauli group
- orthogonal operator basis: can expand \( \rho = \sum_{\mathbf{v}} \chi_\rho(\mathbf{v}) P_{\mathbf{v}} \), where \( \chi_\rho(\mathbf{v}) = 2^{-n} \text{tr}[\rho P_{\mathbf{v}}] \) characteristic function

Qudits: phase space \( \mathbb{F}_d^{2n} \) corresponding to ‘shift’ and ‘clock’ operators:

\[
X |q\rangle = |q + 1 \pmod{d}\rangle \\
Z |q\rangle = e^{2\pi i q/d} |q\rangle
\]
Pauli operators and discrete phase space

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$$Z \ |q\rangle = e^{2\pi i q/d} \ |q\rangle$$
Clifford unitaries

Clifford group: Unitaries $U_C$ such that $P$ Pauli $\Rightarrow U_C P U_C^\dagger \propto$ Pauli.
For qubits, generated by CNOT, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$.

E.g.,

Clifford unitaries = classical dynamics on discrete phase space

$\Rightarrow$ for any symplectic matrix $\Gamma$, exists Clifford $U_\Gamma$ with $U_\Gamma P_x U_\Gamma^\dagger \propto P_\Gamma x$
$\Rightarrow$ conversely, any Clifford unitary is of form $U_C \propto U_\Gamma P_v$

Clifford circuits can be simulated efficiently on a classical computer (Gottesman-Knill)
Stabilizer states

States of the form \( |S\rangle = U_C |0\rangle \otimes^n \).

- computational basis states, maximally entangled states, GHZ states...
- QEC, MBQC, topological order, ...

Equivalently, stabilized by maximal commutative subgroup \( G \) of Paulis:

\[
|S\rangle\langle S| = d^{-n} \sum_{P \in G} P
\]

E.g., \( |00\rangle + |11\rangle \) defined by \( G = \langle XX, ZZ \rangle \).
Stabilizer states

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In terms of discrete phase space: \( G = \{ e^{2\pi i f(v)/d} P_v \mid v \in V \} \), where

- \( V \subseteq \mathbb{F}_d^{2n} \) isotropic: \( [v, w] = 0 \) for all \( v, w \in V \)
- maximal dimension: \( \dim V = n \)

E.g., \( |00\rangle + |11\rangle \) defined by \( G = \langle XX, ZZ \rangle, \ V = \langle (0011), (1100) \rangle \).
Stabilizer codes and quantum error correction

Obtain stabilizer codes if \( V \) isotropic but not of maximal dimension. E.g.,

- 1 → 3 bit flip code is defined by \( ZZI, IZZ \)

\[
\begin{align*}
|\psi\rangle & \quad \bullet \quad \bullet \quad \bullet \quad \bigcirc \quad |\psi\rangle \\
|0\rangle & \quad \bigcirc \quad |0\rangle \\
\end{align*}
\]

\[
E_{\text{bit}}
\]

- 1 → 5 qubit code has stabilizers \( XZZXI, IXZZX, XIXZZ, ZXIXZ \)

Useful properties:

- ‘stabilizers = syndrome’
- encoding and error correction circuits are Clifford
- q. error correction condition only depends on \( V \) (for Pauli errors)!
Wigner function and classical simulation

For odd $d$, every quantum state has a discrete Wigner function:

$$W_\rho(v) = \hat{\chi}_\rho(v) = d^{-2n} \sum_w e^{-2\pi i [v,w]/d} \text{tr}[\rho P_v]$$

- quasi-probability distribution on phase space $\mathbb{F}_d^{2n}$
- Clifford-covariant
- discrete Hudson theorem (Gross): for pure states, $W_\psi \geq 0$ iff stabilizer

Non-negative Wigner function $\Rightarrow$ efficient classical simulation

- Wigner negativity $\text{sn}(\psi) = \sum_v: W_\rho(v) < 0 |W_\rho(v)|$
- resource theory of stabilizer computation, contextuality, …
- see work by Veitch et al, Pashayan et al, Raussendorf et al, Howard-Campbell, …
Derandomization and designs

Randomized constructions often rely on Haar measure. Simple to analyze, often near-optimal – but inefficient!

A unitary $t$-design $\{U_j\}$ has same $t$-th moments as Haar measure on $U(D)$:

$$E_j[(U_j \otimes U_j^\dagger)^\otimes t] = E_{\text{Haar}}[(U \otimes U^\dagger)^\otimes t]$$

A state $t$-design $\{\psi_j\}$ has same $t$-th moments as ‘Haar measure’ on pure states:

$$E_j[|\psi_j\rangle\langle\psi_j|^\otimes t] = E_{\text{Haar}}[|\psi\rangle\langle\psi|^\otimes t]$$

- Clifford unitaries and stabilizer states are 2-design; 3-design for qubits (Küng-Gross, Zhu, Webb)
- many applications: randomized benchmarking, phase retrieval, low-rank matrix recovery, . . .
Schur-Weyl duality for the Clifford group
Schur-Weyl duality

Two *symmetries* that are ubiquitous in quantum information theory:

\[ U \otimes^t |x_1, \ldots, x_t\rangle = U |x_1\rangle \otimes \ldots \otimes U |x_t\rangle \]
\[ R_\pi |x_1, \ldots, x_t\rangle = |x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(t)}\rangle \]

- i.i.d. quantum information: \([\rho^{\otimes t}, R_\pi] = 0\)
- eigenvalues, entropies, \ldots: \(\rho \equiv U \rho U^\dagger\)
- randomized constructions: \(E_{\text{Haar}}[|\psi\rangle\langle\psi|^{\otimes^t}]\)

*Would like a version for Clifford unitaries \(U_C\)!*

\((\mathbb{C}^D)^{\otimes^t}\)
Schur-Weyl duality

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\begin{align*}
U^\otimes t \ket{x_1, \ldots, x_t} &= U \ket{x_1} \otimes \ldots \otimes U \ket{x_t} \\
R_\pi \ket{x_1, \ldots, x_t} &= \ket{x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(t)}}
\end{align*}
\]

Schur-Weyl duality: These actions generate each other's commutant.

- i.i.d. quantum information: \([\rho^\otimes t, R_\pi] = 0\)
- Eigenvalues, entropies, \ldots: \(\rho \equiv U \rho U^\dagger\)
- Randomized constructions: \(E_{\text{Haar}}[\ket{\psi}\bra{\psi}^\otimes t] \propto \sum_{\pi \in S_t} R_\pi\)

Would like a version for Clifford unitaries \(U_C\)!
Our results

“Schur-Weyl duality” for the **Clifford group**: We characterize precisely which operators commute with $U_C^\otimes t$ for all Clifford unitaries $U_C$.

Fewer unitaries $\sim$ larger commutant (more than permutations).

Applications:

- Property testing
- De Finetti theorems with increased symmetry
- Higher moments of stabilizer states
- $t$-designs from Clifford orbits
- Robust Hudson theorem
Permuting blocks

Permutation of $t$ copies of $(\mathbb{C}^d)^\otimes n$:

\begin{align*}
R_\pi &= r_\pi^{\otimes n}, \\
r_\pi &= \sum_x |\pi x\rangle \langle x|
\end{align*}

Here, we think of $\pi$ as $t \times t$-permutation matrix, and $|x\rangle = |x_1, \ldots, x_t\rangle$ is computational basis of $(\mathbb{C}^d)^\otimes t$.

The commutant of $\{U_C^\otimes t\}$ is given by a straightforward generalization...
Schur-Weyl duality for the Clifford group

\[ (\mathbb{C}^d)^\otimes n \]

\[ R_T = r_T^\otimes n, \quad r_T = \sum_{(y,x) \in T} |y\rangle \langle x| \]

Allow all subspaces \( T \subseteq \mathbb{F}_d^{2t} \) that are self-dual codes, i.e. \( y \cdot y' \equiv x \cdot x' \) and of maximal dimension \( t \). Moreover, require \( |y| \equiv |x| \) (for qubits, modulo 4).

**Theorem**

For \( n \geq t - 1 \), the operators \( R_T \) are \( \prod_{k=0}^{t-2} (d^k + 1) \) many linearly independent operators that span the commutant of \( \{ U_C^\otimes t \} \).

*Independent of \( n \) (just like in ordinary Schur-Weyl duality)! Rich algebraic structure (see paper).*
Examples of commutant

Want subspaces $T \subseteq F_d^{2t}$ that are self-dual codes, i.e. $y \cdot y' \equiv x \cdot x'$ and of maximal dimension $t$. Moreover, require $|y| \equiv |x|$ (for qubits, modulo 4).

For qubits, an example is the following code for $t = 4$:

$$T = \text{ran} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

$$R_T = 2^{-n} \left( I \otimes^4 + X \otimes^4 + Y \otimes^4 + Z \otimes^4 \right)$$

The projector $R_T$ commutes with $U_C^4$ for every $n$-qubit Clifford unitary. Central to 4-th moments of multiqubit stabilizer states (Zhu et al, later).
Examples of commutant

Can also obtain subspaces as graphs $T = \{(Ox, x)\}$ of $t \times t$ orthogonal stochastic matrices. Then $R_O = r_O^\otimes n$, $r_O = \sum_x |Ox\rangle \langle x|$ is in commutant.

For qubits, an example is the $6 \times 6$ anti-identity:

$$\overline{id} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

$$R_{\overline{id}} |x_1, \ldots, x_6\rangle = |x_2 + \ldots + x_6, \ldots, x_1 + \ldots + x_5\rangle$$

The unitary $R_{\overline{id}}$ commutes with $U_C^\otimes 6$ for every $n$-qubit Clifford unitary.

The $\{R_O\}$ are symmetries of stabilizer tensor powers $\sim$ de Finetti (later).
Why should the theorem be true? 

\[ R_T = r_T^\otimes n, \quad r_T = \sum_{(y,x) \in T} |y\rangle \langle x| \]

When is \( R_T \) in the commutant? Need that \( T \subseteq \mathbb{F}_2^{2t} \) is...

- **subspace:**
  \[ \text{CNOT}^\otimes t r_T^{\otimes 2} \text{CNOT}^\otimes t = \sum_{(y,x), (y',x') \in T} |y\rangle \langle x| \otimes |y+y'|\langle x+x'| = r_T^{\otimes 2} \]

- **self-dual:**
  \[ H^\otimes t r_T H^\otimes t = \sum_{(y',x') \in T^\perp} |y'\rangle \langle x'| = r_T \]

- **modulo 4:**
  \[ P^\otimes t r_T P^\dagger,^\otimes t = \sum_{(y,x) \in T} i|y| - |x| |y\rangle \langle x| = r_T \]

*Remainder of proof:* Show that \( R_T \)'s linearly independent. Compute dimension of commutant (\#group orbits) & number of subspaces as above (Witt's lemma).
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  \[ (y,x),(y',x') \in T \]

- **self-dual:**
  \[ \text{H}^\otimes t \ r_T \ \text{H}^\otimes t = \sum_{y',x'} |y'\rangle \langle x'| \ 2^{-t} \sum_{(y,x) \in T} (-1)^{y \cdot y' + x \cdot x'} = \sum \ |y'\rangle \langle x'| = r_T \]
  \[ (y',x') \in T^\perp \]

- **modulo 4:** \( P^\otimes t \ r_T \ P_t^\dagger = \sum_{(y,x) \in T} i^{|y|-|x|} |y\rangle \langle x| = r_T \)

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**Remainder of proof:** Show that \( R_T \)'s linearly independent. Compute dimension of commutant (\#group orbits) & number of subspaces as above (Witt's lemma). \( \square \)
Application 1: Higher moments of stabilizer states

Result (t-th moment)

\[ E[|S\rangle\langle S|^\otimes t] \propto \sum_T R_T \]

- When stabilizer states form t-design, reduces to \( \sum_\pi R_\pi \) (Haar average)
- Summarizes all previous results on statistical properties
- ...but works for any t-th moment!

Many applications: Improved bounds for randomized benchmarking (Helsen et al, Bas’ poster!), low-rank matrix recovery (Kueng et al); analytical studies of scrambling in Clifford circuits; toy models of holography (Nezami-W); ...

*We can also write t-th moment as weighted sum of certain CSS codes.*
Application 2: Stabilizer testing

Given $t$ copies of an unknown state in $(\mathbb{C}^d)^\otimes n$, decide if it is a stabilizer state or $\varepsilon$-far from it.

**Idea:** Use the anti-identity. Measure POVM element $\frac{1+R_{\text{id}}}{2}$ on $t = 6$ copies.

**Result**

Let $\psi$ be a pure state of $n$ qubits. If $\psi$ is a stabilizer state then this accepts always. But if $\max_S |\langle \psi | S \rangle|^2 \leq 1 - \varepsilon^2$, acceptance probability $\leq 1 - \varepsilon^2/4$.

- Power of test independent of $n$. Answers q. by Montanaro & de Wolf.
- Similar result for qudits & for testing if blackbox unitary is Clifford.

Why does it work? How to implement?
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Why does it work? How to implement?
Stabilizer testing using Bell difference sampling

Any state $\psi$ can be expanded in Pauli basis$^\dagger$:

$$\psi = \sum_{v} \chi_{\psi}(v) P_v$$

- If pure, then $p_{\psi}(v) = 2^n |\chi_{\psi}(v)|^2$ is a probability distribution.
- If stabilizer state, then support of $p_{\psi}$ is stabilizer group (up to sign).

Key idea: Sample & verify!

How to sample? If $\psi$ is real, can simply measure in Bell basis $(P_v \otimes I) |\Phi^+\rangle$ (Bell sampling; Montanaro, Zhao et al).

$^\dagger$recall $P_v = P_{v_1} \otimes \ldots \otimes P_{v_n}$ where $P_{00} = I$, $P_{01} = X$, $P_{10} = Z$, $P_{11} = Y$
Stabilizer testing using Bell difference sampling

In general, need to take ‘difference’ of two Bell measurement outcomes:

- Fully transversal circuit, only need coherent two-qubit operations.
- Circuit is equivalent to measuring the anti-identity!

Proof of converse uses uncertainty relation.

How to test stabilizer rank?
Application 3: Stabilizer de Finetti theorems

Any tensor power $|\psi\rangle \otimes^t$ has $S_t$-symmetry. De Finetti theorems provide ‘partial’ converse: If $|\Psi\rangle$ has $S_t$-symmetry, $\Psi_s \approx \int d\mu(\psi) \psi^{\otimes s}$ for $s \ll t$.

Stabilizer tensor powers have increased symmetry:

$$R_O |S\rangle^{\otimes^t} = |S\rangle^{\otimes^t} \quad \text{for all orthogonal and stochastic } O$$

Result

Assume that $|\Psi\rangle \in (\mathbb{C}^d)^{\otimes^n} \otimes^t$ has this symmetry. Then:

$$\|\Psi_s - \sum_S p_S |S\rangle S^{\otimes^s}\|_1 \lesssim d^{2n(n+2)} d^{-(t-s)/2}$$

- Approximation is exponentially good, by bona fide stabilizer states.
- Similar to Gaussian de Finetti (Leverrier et al). Applications to QKD?

Can reduce symmetry requirements at expense of goodness.
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$$R_O \left| S \right\rangle^\otimes t = \left| S \right\rangle^\otimes t$$

for all orthogonal and stochastic $O$.

Result

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Application 4: \( t \)-designs from Clifford orbits

When \( t > 2 \) or \( 3 \) (qubits), stabilizer states fail to be \( t \)-design. Yet, hints in the literature that this failure is relatively \textit{graceful} (Zhu \textit{et al}, Nezami-W). E.g., Clifford orbit of non-stabilizer qutrit states can be 3-design!

We prove in general:

\textbf{Result}

For every \( t \), there exists ensemble of \( N = N(d, t) \) many fiducial states in \( (\mathbb{C}^d)^\otimes n \) such that corresponding Clifford orbits form \( t \)-design.

- Number of fiducials does not depend on \( n \! \)
- Efficient construction?
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- Number of fiducials does not depend on $n$!
- Efficient construction?
Application 5: Robust Hudson theorem

Recall: For odd $d$, every quantum state has a discrete Wigner function:

$$W_\rho(v) = d^{-2n} \sum_w e^{-2\pi i [v,w]/d} \text{tr}[\rho P_v]$$

- Quasi-probability distribution on phase space $\mathbb{R}^{2n}_d$
- **Discrete Hudson theorem**: For pure states, $W_\psi \geq 0$ iff $\psi$ stabilizer
- **Wigner negativity** $\text{sn}(\psi) = \sum_v: W_\rho(v) < 0 |W_\rho(v)|$: monotone in resource theory of stabilizer computation; witness for contextuality

**Result (Robust Hudson)**

There exists a stabilizer state $|S\rangle$ such that $|\langle S|\psi\rangle|^2 \geq 1 - 9d^2 \text{sn}(\psi)$. 
Application 6: Typical entanglement of stabilizer states

Tripartite stabilizer states decompose into EPR and GHZ entanglement:

\[ g = O(1) \text{ w.h.p.} \]

▶ can distill \( \frac{1}{2} I(A : B) \) EPR pairs
▶ mutual information is entanglement measure
▶ generalizes result by Leung & Smith (qubits, single tensor)

How about typical stabilizer states? Or even tensor networks?

Result (Nezami-W)

In random stabilizer tensor network states: \( g = O(1) \text{ w.h.p.} \)
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Result (Nezami-W)

In random stabilizer tensor network states: \( g = O(1) \) w.h.p.

- can distill \( \approx \frac{1}{2} I(A : B) \) EPR pairs
- mutual information is entanglement measure
- generalizes result by Leung & Smith (qubits, single tensor)
Bounding the amount of GHZ entanglement

\[ I(A : B) = 2c + g \]

Diagnose via third moment of partial transpose:

\[ g \log d = S(A) + S(B) + S(C) + \log \text{tr}(\rho_{AB}^T)^3 \]

Compute via replica trick: For single stabilizer state

\[ \text{tr}(\rho_{AB}^T)^3 = \text{tr} |S\rangle \langle S|_{ABC}^\otimes^3 \left(R_{\zeta,A} \otimes R_{\zeta^{-1},B} \otimes R_{\text{id},C} \right) \]

where \( \zeta = (1 2 3) \) three-cycle, hence

\[ \mathbb{E}[\text{tr}(\rho_{AB}^T)^3] \propto \sum_T (\text{tr} r_T r_\zeta)^{n_A} (\text{tr} r_T r_{\zeta^{-1}})^{n_B} (\text{tr} r_T r_{\text{id}})^{n_C} \]

Similarly for tensor networks \( \sim \) classical statistical model!
Bounding the amount of GHZ entanglement

\[ I(A : B) = 2c + g \]

Diagnose via third moment of partial transpose:

\[ g \log d = S(A) + S(B) + S(C) + \log \text{tr}(\rho_{AB}^T)^3 \]

Compute via replica trick: For single stabilizer state

\[ \text{tr}(\rho_{AB}^T)^3 = \text{tr} |S\rangle\langle S|_{ABC}^\otimes (R_{\zeta,A} \otimes R_{\zeta^{-1},B} \otimes R_{\text{id},C}) \]

where \( \zeta = (1 2 3) \) three-cycle, hence

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For simplicity, assume $A$, $B$, $C$ each $n$ qubits.

$$\mathbb{E}[g] \leq 3n + \log \mathbb{E}[\text{tr}(\rho_{AB}^{T_B})^3]$$

Since qubit stabilizers are three-design:

$$\mathbb{E}[\text{tr}(\rho_{AB}^{T_B})^3] = \sum_{\pi \in S_3} 2^{-n} \left( d(\zeta, \pi) + d(\zeta^{-1}, \pi) + d(\text{id}, \pi) \right)$$

where $d(\pi, \tau) = \text{minimum number of swaps needed for } \pi \leftrightarrow \tau$. Thus:

$$\mathbb{E}[\text{tr}(\rho_{AB}^{T_B})^3] \leq 3 \cdot \underbrace{2^{-3n}}_{\text{swaps}} + 3 \cdot \underbrace{2^{-4n}}_{\text{id, } \zeta, \zeta^{-1}} \Rightarrow \mathbb{E}[g] \lesssim \log 3 \quad \square$$

For $d > 2$, $\{T\} = \{\text{even}\} \cup \{\text{odd}\}$. Calculation completely analogous!
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Pauli & Clifford unitaries, stabilizer states in $(\mathbb{C}^d)^\otimes n$:
- best understood via discrete phase space $\mathbb{F}_d^{2n}$

Schur-Weyl duality for the Clifford group:
- clean algebraic description in terms of self-dual codes
- resolve open question in quantum property testing
- new tools for stabilizer states: moments, de Finetti, Hudson, ...