Q. Entropies and Representation Theory

joint work with M. Christandl, B. Sahinoglu

1 Introduction

$(p_k)_{k=1,\dots,d}$ probability distribution

Shannon entropy:

$$H(p) := -\sum_{k=1}^{d} p_k \cdot \log(p_k) \qquad 0 \cdot \log 0 = 0$$

→ $=0$ iff p deterministic, else >0, max. if p uniform

→ optimal rate of compression, info. theory, stat. mechanics

Quantum Mechanics:

- state of q. system is described by density operator $\rho \geq 0$, $\operatorname{tr} \rho = 1$ on Hilbert space $\mathcal{H}$
- von Neumann entropy:

$$S(\rho) := -\operatorname{tr} \rho \cdot \log \rho = -\sum_k p_k \cdot \log p_k$$

eigenvalues of ρ

→ q. info theory, q. stat. mechanics, ...

- composite $\underbrace{\text{systems}}_{\text{of dist. particles}}$ are described by tensor-product of HS, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

- if ρ_{AB} is the state of the comp. system AB: subsystem A is described by the <u>reduced density matrix</u> ρ_A, defined by

$$\mathrm{tr}(\rho_A X_A) = \mathrm{tr}(\rho_{AB} \cdot X_A \otimes \mathbb{1}_B)$$

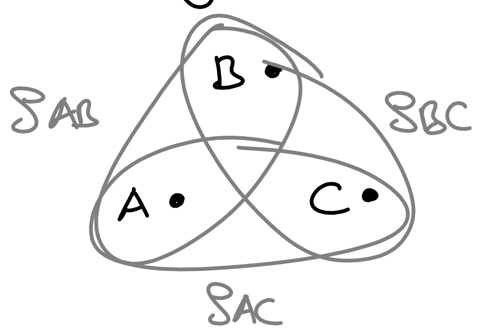
expectation values

for all (Herm.) operators X_A on $\mathcal{H}_A$.

<u>Some fundamental questions:</u>

Given a tensor-product HS $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

① When are given density matrices compatible?



$\exists \rho_{ABC}$?

<u>Quantum Marginal Problem</u>

② What is the relation between the entropies $S(\rho_{ABC})$, $S(\rho_{AB})$, ... ? $\longrightarrow$ <u>Entropy Inequalities</u>

Only known constraints:

- $S(\rho) \geq 0$
- $\boxed{S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B)}$ THIS TALK

(<u>Strong Subadditivity</u>, Lieb–Ruskai)

- $S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_A) + S(\rho_C)$

(<u>Weak Monotonicity</u>)

② Eigenvalues & Representation Theory

ρ density operator on $\mathcal{H} = \mathbb{C}^d$

$$\to \underbrace{\boxed{\rho^{\otimes N}}}_{\text{N "i.i.d." copies of the state } \rho} \text{ on } \mathcal{H}^{\otimes N} = \underbrace{\boxed{(\mathbb{C}^d)^{\otimes N}}}_{\text{representation of } U(d) \times S_N}$$

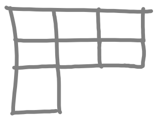
Schur-Weyl duality:

$$(\mathbb{C}^d)^{\otimes N} \cong \bigoplus_{\lambda} V_\lambda^d \otimes [\lambda]_N$$

Young diagram $\lambda \in \mathbb{N}^d_{\geq 0}$, $\lambda_i \geq \lambda_{i+1}$, $\sum_i \lambda_i = N$

GL(d)-irrep

S_N-irrep

3 parts  7 boxes

$\bar{\lambda} := \lambda / N$

LOOKS LIKE "QUANTIZ." OF SPECTRUM

$\rho^{\otimes N}$ is permut.-inv. $\rightsquigarrow$ By Schur's lemma:

$$\underbrace{\rho^{\otimes N}}_{tr = 1} \cong \bigoplus_{\lambda} p_\lambda \cdot \underbrace{\rho_\lambda^d \otimes \frac{\mathbb{1}[\lambda]}{\dim [\lambda]}}_{tr = 1}$$

probability distribution

What does the distribution p_λ look like?

$$p_\lambda = tr(P_\lambda \rho^{\otimes N} P_\lambda) = \boxed{\chi_\lambda^d(\rho)} \cdot \dim [\lambda]$$

GL char.
$\to$ can extend to ρ

Asymptotics:

(Character $GL(d)$-irrep V_λ

- WLOG $\varrho = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_d \end{pmatrix}$

Let $T(d) := \{\text{diag. matrices}\} \subseteq GL(d)$.

Irreps are 1d, labeled by weight $w \in \mathbb{N}_{\geq 0}^d$, such that $\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_d \end{pmatrix}$ acts by $z_1^{w_1} \cdots z_d^{w_d}$.

poly irreps

Fact: Only weights $w \preceq \lambda$ occur in V_λ^d.

$$w_1 \leq \lambda_1, \ w_1 + w_2 \leq \lambda_1 + \lambda_2, \ldots$$

$$a^{w_1} b^{w_2} = \left(\frac{a}{b}\right)^{w_1} b^{w_1+w_2} = \ldots$$

$$\Rightarrow r_1^{w_1} \cdots r_d^{w_d} \leq r_1^{\lambda_1} \cdots r_d^{\lambda_d}$$

$$\Rightarrow \chi_\lambda^d(\varrho) \leq \underbrace{\dim(V_\lambda^d) \cdot r_1^{\lambda_1} \cdots r_d^{\lambda_d}}_{\text{leading-order term in WCF (Verma module!)}}$$

$$\leq \operatorname{poly}(N) \cdot r_1^{\lambda_1} \cdots r_d^{\lambda_d}$$

e.g. $(N+1)^{d(d-1)/2}$

$$= \operatorname{poly}(N) \cdot e^{N \cdot \sum_k \bar\lambda_k \cdot \log r_k}$$

Stirling: $N! \sim \sqrt{2\pi N} \, (N/e)^N$

- $\dim [\lambda] \leq \binom{N}{\lambda}$

$$\lesssim e^{N \cdot \log N - \sum_k \lambda_k \cdot \log \lambda_k}$$

$$= e^{-N \cdot \boxed{\sum_k \bar\lambda_k \cdot \log \bar\lambda_k}}$$

$= H(\bar\lambda)$

Summary:

Ⓐ $p_\lambda = \operatorname{tr}(P_\lambda \rho^{\otimes N}) \leq \operatorname{poly}(N) \cdot e^{-N \boxed{\sum_k \bar\lambda_k \cdot \log \frac{\bar\lambda_k}{r_k}}}$

relative entropy

Pinsker

$$\leq \operatorname{poly}(N) \cdot e^{-N \|\bar\lambda - r\|_1^2 / 2}$$

[Keyl-Werner, Hayashi-Matsumoto, Chr.-Mitchison]

$\Longrightarrow$ $\rho^{\otimes N}$ is sharply concentrated on those components with $\bar\lambda \approx r = \operatorname{spec} \rho$ (spectrum estimation)

More precisely, if we set $\tilde{P}^N := \bigoplus_{\bar\lambda \approx_\varepsilon r} P_\lambda$ then

$$\operatorname{tr} \rho^{\otimes N} \tilde{P}^N \geq 1 - \operatorname{poly}'(N) \cdot e^{-N\varepsilon^2/2} \longrightarrow 1$$

even if we let $\varepsilon \to 0$ as $N \to \infty$ (slowly).

Ⓑ $\frac{1}{N} \cdot \log \dim [\lambda] \approx H(\bar\lambda)$ (by a more careful analysis)

$\Longrightarrow$ Shannon entropy determines growth rate of S_N-irreps

Together:

Typical subspace for $\rho^{\otimes N}$ grows as $H(r) = S(\rho)$

$\to$ q. data compression, ...

③ Subadditivity $\quad S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$

Let ρ_{AB} be a q. state on $\mathbb{C}^a \otimes \mathbb{C}^b$.

Define $\tilde{P}_{AB}, \tilde{P}_A, \tilde{P}_B$ as in Ⓐ:

- $\mathrm{tr}(\rho_{AB}^{\otimes N} \tilde{P}_{AB}) \to 1$
- $\mathrm{tr}(\rho_A^{\otimes N} \tilde{P}_A) = \mathrm{tr}(\rho_{AB}^{\otimes N} (\tilde{P}_A \otimes \mathbb{1}_B)) \to 1$
- $\mathrm{tr}(\rho_B^{\otimes N} \tilde{P}_B) = \mathrm{tr}(\rho_{AB}^{\otimes N} (\mathbb{1}_A \otimes \tilde{P}_B)) \to 1$

$$\xRightarrow{\text{union bound}} \mathrm{tr}(\rho_{AB}^{\otimes N} \tilde{P}_{AB} (\tilde{P}_A \otimes \tilde{P}_B)) > 0$$

In particular:

$$\boxed{P_{\lambda_{AB}} (P_{\lambda_A} \otimes P_{\lambda_B}) \neq 0} \qquad (*)$$

for (a seq. of) Young diagrams $\lambda_{AB}, \ldots$ such that $\bar{\lambda}_{AB} \approx r_{AB} = \mathrm{spec}(\rho_{AB}), \ldots$

What does $(*)$ mean?

$$(\mathbb{C}^{ab})^{\otimes N} = (\mathbb{C}^a)^{\otimes N} \otimes (\mathbb{C}^b)^{\otimes N}$$

$$= \left(\bigoplus_{\lambda_A} V^a_{\lambda_A} \otimes [\lambda_A] \right) \otimes \left(\bigoplus_{\lambda_B} V^b_{\lambda_B} \otimes [\lambda_B] \right)$$

$$= \bigoplus_{\lambda_A, \lambda_B} V^a_{\lambda_A} \otimes V^b_{\lambda_B} \otimes \underbrace{[\lambda_A] \otimes [\lambda_B]}_{S_N \times S_N}$$

$$= \underbrace{\bigoplus_{\lambda_A, \lambda_B} \bigoplus_{\lambda_{AB}} V^a_{\lambda_A} \otimes V^b_{\lambda_B} \otimes g^{\lambda_{AB}}_{\lambda_A, \lambda_B} \cdot [\lambda_{AB}]}_{\text{range of } (P_{\lambda_A} \otimes P_{\lambda_B}) P_{\lambda_{AB}} \,!} \quad S_N$$

In particular, ⊛ shows:

$$[\lambda_{AB}] \subseteq [\lambda_A] \otimes [\lambda_B]$$

Can also get this from alg. & sympl. geometry / Kirwan polytope / ...

$$\Longrightarrow \quad \dim\,[\lambda_{AB}] \leq \dim\,[\lambda_A] \cdot \dim\,[\lambda_B]$$

$$\xRightarrow[\frac{1}{N}\log(\cdot\cdot)]{\boxed{B}} \quad S(\rho_{AB}) = H(r_{AB})$$

$$\leq H(r_A) + H(r_B) = S(\rho_A) + S(\rho_B).$$

[Chr.–Mitchison]

<u>Summary</u>:

Dimension relation gives inequality

$$\begin{tikzcd} & \rho_{AB} \arrow[dl] \arrow[dr] & \\ \rho_A & & \rho_B \end{tikzcd} \quad \Longrightarrow \quad \begin{array}{c} [\lambda_{AB}] \\ \cap\!\shortparallel \\ [\lambda_A] \otimes [\lambda_B] \end{array} \quad \Longrightarrow \quad \begin{array}{l} S(\rho_{AB}) \leq S(\rho_A) \\ \qquad\qquad + S(\rho_B) \end{array}$$

Quantum state is <u>witness</u> for inclusion

⌈Can also be understood as <u>semiclassical limit</u>:

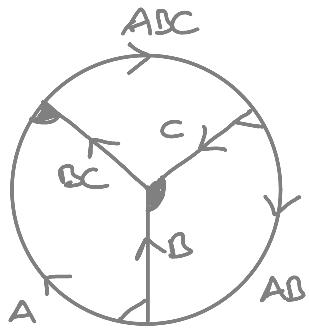
$$GL(a) \times GL(b) \xrightarrow{\otimes} GL(ab)$$

by Schur–Weyl duality. ⌋

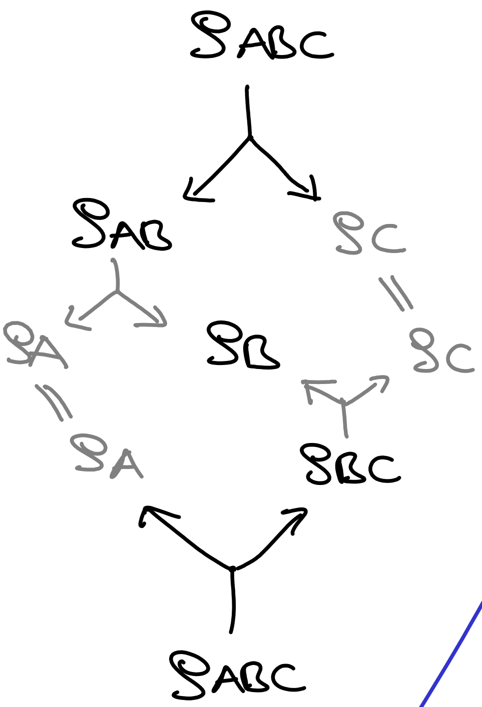
④ Strong Subadditivity

$$S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B)$$

Need to control many spectra!



Ansatz:



$\Longrightarrow$

Two S_N-equ. (isom.) embeddings

$$\Phi: [\lambda_{ABC}] \subseteq [\lambda_{AB}] \otimes [\lambda_C] \subseteq ([\lambda_A] \otimes [\lambda_B]) \otimes [\lambda_C]$$

and

$$\Psi: [\lambda_{ABC}] \subseteq [\lambda_A] \otimes [\lambda_{BC}] \subseteq [\lambda_A] \otimes ([\lambda_B] \otimes [\lambda_C])$$

$\rightsquigarrow$ measure "overlap" by

$$\Psi^* \circ \Phi = C \cdot \mathrm{id}_{[\lambda]}$$

"recoupling coefficient"

(linear map! multiplicities!)

$$C \leq \frac{\dim[\lambda_{AB}] \cdot \dim[\lambda_{BC}]}{\dim[\lambda_{ABC}] \cdot \dim[\lambda_B]} \cdot \mathrm{poly}(N)$$

Quantum state ρ_{ABC} is witness that recoupling coeffs are large

$$C \geq 1/\mathrm{poly}(N)$$

$$\boxed{B} \Longrightarrow_{\frac{1}{N}\log(\cdot)} S(\rho_{AB}) + S(\rho_{BC}) \geq S(\rho_{ABC}) + S(\rho_B)$$

Open Questions:

- How to control more spectra?
- Do we really need to understand a sequence of coefficients?
- Vision: "Equivalence"

  entropy ineqs $\overset{?}{\longleftrightarrow}$ representation-theoretic data

  (as in the classical case)
- New inequalities for the v. N. entropy?

  (as in the classical case)