A primer on the method of hypergraph containers

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KdVI General Mathematics Colloquium
Extremal Combinatorics

"Complete disorder is impossible" – Theodore S. Motzkin

How large can an object be before containing some specific (structured) sub-object?
Extremal Combinatorics

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The party theorem:

In any group of six people, either three people mutually know each other or three people mutually don't know each other.*
Extremal Combinatorics

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Extremal Combinatorics

How large can an object be before containing some specific (structured) sub-object? A graph $K_6$.

The Party Theorem: In any group of six people, either three people mutually know each other or three people mutually don't know each other.*
Extremal Combinatorics

Abstraction leads to natural follow-up questions:

- For every \( k, r \), does there exist \( n \) s.t. two coloring \( K_n \) gives red \( K_k \) or blue \( K_r \)?

Recall: We just saw that for \( k = r = 3 \), \( n = 6 \) suffices

\( k = r = 4 \Rightarrow n = 18 \)

\( k = r = 5 \) ? \( 43 \leq n \leq 48 \)

"a 2-edge coloring of the complete graph \( K_6 \)"
Extremal Combinatorics

Abstraction leads to natural follow-up questions:

- For every $k, r$, does there exist $n$ s.t. two coloring $K_n$ gives red $K_k$ or blue $K_r$?
- What about more than two colors?

"a 2-edge coloring of the complete graph $K_6"
Extremal Combinatorics

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- For every $k, r$, does there exist $n$ s.t. two coloring $K_n$ gives red $K_k$ or blue $K_r$?
- What about more than two colors?
- What about multi-colored objects?

"a 2-edge coloring of the complete graph $K_6"
(Hyper)graphs

- A hypergraph is a pair $\mathcal{H} = (V, E)$
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- $V$ the set of vertices or nodes
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- A hypergraph is a pair $\mathcal{H} = (V, E)$
- $V$ the set of vertices or nodes
- $E \subseteq 2^V$ the set (system) of (hyper)-edges
  - powerset of $V$
(Hyper)graphs

• A hypergraph is a pair $\mathcal{H} = (V, E)$

• $V$ the set of vertices or nodes

• $E \subseteq 2^V$ the set (system) of (hyper)-edges

• If $|e| = k \in \mathbb{N}$ for all edges $e \in E$, we call $\mathcal{H}$ $k$-uniform

usually fixed, but may depend on $|V|$ as well
(Hyper)graphs

- A hypergraph is a pair \( \mathcal{H} = (V, E) \)
- \( V \) the set of vertices or nodes
- \( E \subseteq 2^V \) the set (system) of (hyper)-edges

- If \( |e| = k \in \mathbb{N} \) for all edges \( e \in E \), we call \( \mathcal{H} \) \( k \)-uniform

- A 2-uniform hypergraph is called graph
(Induced) sub-hypergraphs and independent sets

Let $\mathcal{H} = (V, E)$ be a hypergraph.

- A hypergraph $\mathcal{F} = (V', E')$ is a subhypergraph of $\mathcal{H}$ if $V' \subseteq V$ and $E' \subseteq E$. 
(Induced) sub-hypergraphs and independent sets

Let $\mathcal{H} = (V, E)$ be a hypergraph.

- A hypergraph $F = (V', E')$ is a subhypergraph of $\mathcal{H}$ if $V' \subseteq V$ and $E' \subseteq E$.

- It is an induced subhypergraph if $E' = 2^{V'} \cap E$; every edge $e \in E$ with $e \subseteq V'$ is in $E'$. 
(Induced) sub-hypergraphs and independent sets

Let \( \mathcal{H} = (V, E) \) be a hypergraph.

- A hypergraph \( \mathcal{F} = (V', E') \) is a subhypergraph of \( \mathcal{H} \) if \( V' \subseteq V \) and \( E' \subseteq E \).
- It is an induced subhypergraph if \( E' = 2^{V'} \cap E \).
- For \( V' \subseteq V \), define \( \mathcal{H}[V'] = (V', 2^{V'} \cap E) \).

the induced subhypergraph on \( V' \)
(Induced) sub-hypergraphs and independent sets

Let $\mathcal{H} = (V, E)$ be a hypergraph.

- A hypergraph $\mathcal{F} = (V', E')$ is a subhypergraph of $\mathcal{H}$ if $V' \subseteq V$ and $E' \subseteq E$.

- It is an induced subhypergraph if $E' = 2^{V'} \cap E$.

- For $V' \subseteq V$, define $\mathcal{H}[V'] = (V', 2^{V'} \cap E)$.

- $I \subseteq V$ is an independent set in $\mathcal{H}$ if $\mathcal{H}[I] = (I, \emptyset)$. 


a hypergraph

a graph
a hypergraph

two independent sets

a graph
Questions about independent sets

Given a specific hypergraph \( \mathcal{H} \) (or maybe a family \((\mathcal{H}_n)_n\))

- What is the largest size of an independent set? (Called the independence number \( \alpha(\mathcal{H}) \))
Questions about independent sets

Given a specific hypergraph $H$ (or maybe a family $(H_n)_n$)

- What is the largest size of an independent set?

- How many independent sets does $H$ have?

NB: Trivial lower bound: $2^{\alpha(H)}$

Often not far from the truth!
Questions about independent sets

Given a specific hypergraph $H$ (or maybe a family $(H_n)_n$)

- What is the largest size of an independent set?

- How many independent sets does $H$ have?

- What is the structure of a typical independent set?

given one uniformly at random
Motivation

Why do we care? Hypergraphs naturally allow encoding of "forbidden substructure" problems: complete graph

- Triangle-free graphs -
  - vertices: edges of \( K_n \)
  - hyperedges: "triangles" \( \{i,j,k\} \), \( \{i,j,k\} \), \( \{i,j,k\} \), \( \{i,j,k\} \)
Motivation

Why do we care? Hypergraphs naturally allow encoding of "forbidden substructure" problems:

- Triangle-free graphs

Vertices: edges of \( K_n \) \{12, 13, 23\}

Hyperedges: "triangles" \{123, 134, 243\}

The hypergraph encoding triangles in \( K_4 \)
Motivation

Why do we care? Hypergraphs naturally allow encoding of "forbidden substructure" problems:

- Triangle-free graphs - hyperedges: "triangles" \{i,j,k\} vertices: edges of \(K_n\) \{i<j\}
- \(k\)-term arithmetic progressions - hyperedges: \(\{x, x + d, \ldots, x + (k-1)d\}\) vertices: \(\{x\} = \{1, \ldots, n\}\)
Motivation

Why do we care? Hypergraphs naturally allow encoding of "forbidden substructure" problems:

- Triangle-free graphs — hyperedges: "triangles" \{i<j<k\}, \{i<j,k\}, \{i<k,j\}
- \(k\)-term arithmetic progressions — hyperedges: \{x, x+d, ..., x+(k-1)d\}

\(\Rightarrow\) Classical results of Mantel (3's) and Szemerédi (3-APs) are about independence number \(\Delta(H)\) of some hypergraph \(H\).
But...

Do we gain anything by encoding a problem as independent sets in hypergraphs?
The method of hypergraph containers

**Idea:** If the edges of a hypergraph $\mathcal{H}=(V,E)$ are distributed "evenly enough", then there exists a small family $\mathcal{C}$ of subsets $C \subseteq V$ ("containers") of $\mathcal{H}$ such that:
The method of hypergraph containers

Idea: If the edges of a hypergraph $\mathcal{H} = (V, E)$ are distributed "evenly enough", then there exists a small family $\mathcal{C}$ of subsets $C \subseteq V$ ("containers") of $\mathcal{H}$ such that:

(i) Every independent set $I$ of $\mathcal{H}$ is contained in some container $C \in \mathcal{C}$. 
The method of hypergraph containers

Idea: If the edges of a hypergraph $\mathcal{H}=(V,E)$ are distributed "evenly enough", then there exists a small family $\mathcal{C}$ of subsets $C \subseteq V$ ("containers") of $\mathcal{H}$ such that:

(i) Every independent set $I$ of $\mathcal{H}$ is contained in some container $C \in \mathcal{C}$.

(ii) For every $C \in \mathcal{C}$, the induced subhypergraph $\mathcal{H}[C]$ has few edges.
The method of hypergraph containers

How does this help with e.g. counting?
The method of hypergraph containers

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We want to count independent sets by union bounding.
The method of hypergraph containers

• How does this help with e.g. counting?
• We want to count independent sets by union bounding

\# \{ \text{ind. sets of } \mathcal{H} \} = \sum_{\mathcal{E} \in \mathcal{E}} \# \{ \text{ind. sets of } \mathcal{H} \text{ in } \mathcal{E} \}
The method of hypergraph containers

• How does this help with e.g. counting?
• We want to count independent sets by union bounding \( \Rightarrow \)
  \[ \# \{ \text{ind. sets of } \mathcal{A} \} = \sum_{C \in \mathcal{E}} \# \{ \text{ind. sets of } \mathcal{A} \text{ in } C \} \]
• Extreme (unhelpful) choices are \( \mathcal{E} = \{ \cup (\emptyset, \emptyset) \} \)
  and \( \mathcal{E} = \{ I \in V : I \ \text{ind.} \} \)
  \[ \uparrow \text{ no control over summands} \]
  \[ \uparrow \text{ no control over summands} \]
The method of hypergraph containers

• How does this help with e.g. counting?
• We want to count independent sets by union bounding

\[ \# \{ \text{ind. sets of } \mathcal{H} \} = \sum_{C \in \mathcal{E}} \# \{ \text{ind. sets of } \mathcal{X} \text{ in } C \} \]

• Extreme (unhelpful) choices are \( E = \{ U(\mathcal{A}\mathcal{E}) \} \) and \( E = \{ I \in V : I \text{ ind.} \} \)
• Our containers represent sweet spot!
The method of hypergraph containers

- For $A \subseteq V(\mathcal{X})$, define $d_{\mathcal{X}}(A) = \# \text{ edges } F \in E(\mathcal{X}) \text{ with } A \subseteq F$
- For $l \in \mathbb{N}$, define $\Delta_l(\mathcal{X}) = \max \{ d_{\mathcal{X}}(A) : A \subseteq V(\mathcal{X}), |A| = l \}$

$d_{\mathcal{X}}$ the degree

$\Delta_l$ the maximum $l$-degree
The method of hypergraph containers

- For $A \subseteq V(\mathcal{H})$, define $d_{\mathcal{H}}(A) = \# \text{edges } F \in E(\mathcal{H})$ with $A \subseteq F$
- For $\ell \in \mathbb{N}$, def $\Delta_{\ell}(\mathcal{H}) = \max \{ d_{\mathcal{H}}(A) : A \subseteq V(\mathcal{H}), |A| = \ell \}$

The container lemma: Let $k \in \mathbb{N}$ and $\delta = 2^{-\kappa(k+1)}$. If $\mathcal{H}$ is $k$-uniform that satisfies

$$\Delta_{\ell} \leq \left( \frac{b}{\nu(\mathcal{H})} \right)^{2-\ell} \frac{e(\mathcal{H})}{r}$$

for some $b, r \in \mathbb{N}$ and all $\ell \in \{1, \ldots, k\}$

NB: $e(\mathcal{H}) = \# \text{edges}$

$v(\mathcal{H}) = \# \text{vertices}$
The method of hypergraph containers

- For $A \subseteq V(\mathcal{H})$, define $d_{\mathcal{H}}(A) = \# \text{ edges } F \in E(\mathcal{H}) \text{ with } A \subseteq F$

- For $l \in \mathbb{N}$, define $\Delta_l(\mathcal{H}) = \max \{ d_{\mathcal{H}}(A) : A \subseteq V(\mathcal{H}), |A| = l^2 \}$

The container lemma: Let $k \in \mathbb{N}$ and $\delta = 2^{-k(k+1)}$. If $\mathcal{H}$ is $k$-uniform that satisfies

$$\Delta_l = \left( \frac{\delta}{u(\mathcal{H})} \right)^{l-1} \frac{e(\mathcal{H})}{r}$$

for some $b, r \in \mathbb{N}$ and all $l \in \{1, \ldots, k\}$, then there exists $C \subseteq 2^{V(\mathcal{H})}$ and $f : 2^{V(\mathcal{H})} \rightarrow C$ s.t.

(i) for every ind. set $I$ of $\mathcal{H}$, exists $S \subseteq I \subseteq f(S)$ with $|S| \leq (k-1)b$
The method of hypergraph containers

- For \( A \subseteq V(H) \), define \( d_{\mathcal{H}}(A) = \# \text{ edges } F \in E(H) \text{ with } A \subseteq F \).
- For \( k \in \mathbb{N} \), define \( \Delta_k(H) = \max \{ d_{\mathcal{H}}(A) : A \subseteq V(H), |A| = k^2 \} \).

The containers lemma: Let \( k \in \mathbb{N} \) and \( \delta = 2^{-k(k+1)} \). If \( H \) is \( k \)-uniform that satisfies
\[
\Delta_k \leq \left( \frac{b}{\binom{\mathbb{N}}{k+1}} \right)^{k-1} \frac{e(H)}{r}
\]
for some \( b, r \in \mathbb{N} \) and all \( k \in \{1, \ldots, k^2 \} \), then there exists \( C \subseteq 2^{V(H)} \) and \( f : 2^{V(H)} \rightarrow C \) s.t.

(i) for every ind. set \( I \) of \( H \), exists \( S \subseteq I \subseteq f(S) \) with \( |S| \leq (k-1)b \).
(ii) \( |C| \leq 2^{V(H)} - 5r \) for every \( C \in C \).
Example triangle-free graphs: The container family

Goal: Obtain containers for n-vertex 4-free graphs

Recall: Our hypergraph family $H_n$ has vertices $\{i,j\}$ with $1 \leq i < j \leq n$ and edges $\{\{i,j\}, \{i,k\}, \{j,k\}\}$ with $1 \leq i, j, k \leq n$.
Example triangle-free graphs: The container family

Goal: Obtain containers for $n$-vertex $\Delta$-free graphs

We see:

1. $\Delta_1(\mathcal{H}_n) = n - 2 = \frac{3e(\mathcal{H})}{\nu(\mathcal{H})}$
2. $\Delta_2(\mathcal{H}_n) = \Delta_3(\mathcal{H}) = 1$

- Fixing a single vertex of $\mathcal{H}_n$:
  - edge of $K_n$
- Fixing two vertices that do not lie in a common hyperedge:
- Two vertices that achieve the max 2-degree:
Example triangle-free graphs: The container family

Goal: Obtain containers for $n$-vertex 4-free graphs

We see:

i) $\Delta_1(\mathcal{H}_n) = n - 2 = \frac{3e(\mathcal{H})}{v(\mathcal{H})}$

ii) $\Delta_2(\mathcal{H}_n) = \Delta_3(\mathcal{H}) = 1$

$\Rightarrow$ Can choose $\tau = \frac{v(\mathcal{H}_n)}{3}$ and $b = \frac{v(\mathcal{H})^{3/2}}{\sqrt{3e(\mathcal{H})}}$

upper bound from

$\Delta_1 \leq \frac{e(\mathcal{H})}{\tau}$

lower bound from

$\Delta_3 \leq \frac{b^2}{v} \cdot \frac{e}{\tau}$ when using

$\tau = \frac{v}{3}$. 
Example triangle-free graphs: The container family

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We see:

i) $\Delta_1(\mathcal{H}_n) = n - 2 = \frac{3e(\mathcal{H})}{\nu(\mathcal{H})}$

ii) $\Delta_2(\mathcal{H}_n) = \Delta_3(\mathcal{H}) = 1$

$\Rightarrow$ Can choose $r = \frac{\nu(\mathcal{H}_n)}{3}$ and $b = \frac{\nu(\mathcal{H})^{3/2}}{\sqrt{3e(\mathcal{H})}}$

Result:

- "fingerprints" of size $\leq \sqrt{\frac{4\nu^3}{3e}} \approx n^{3/2}$
- containers have size $\leq (1 - \frac{\Delta}{3})\nu$

Problem: $\mathcal{H}[C_3]$ might still contain many edges
Example triangle-free graphs: The container family

Solution: As long as $|E[C]|$ has more than $\varepsilon e(|X|)$ edges, re-apply lemma with $r' = \varepsilon r$!
Example triangle-free graphs: The container family

Solution: As long as \( V[C] \) has more than \( \varepsilon e(H) \) edges, re-apply lemma with \( r' = \varepsilon r \). Works, since

\[
\Delta_e(V[C]) \leq \Delta_e(H) \leq \left( \frac{b}{v(H)} \right)^{l-1} \frac{\varepsilon e(H)}{\varepsilon r} = \frac{e(H)}{r}
\]

\[
\leq \left( \frac{b}{v(V[C])} \right)^{l-1} \cdot \frac{e(H)}{r'}
\]

we use \( e(V[C]) \geq \varepsilon e(H) \)

and \( v(V) \geq v(V[C]) \)
Example triangle-free graphs: The container family

Solution: As long as $\mathcal{H}[C]$ has more than $\varepsilon e(\mathcal{H})$ edges, re-apply lemma with $\tau' = \varepsilon r$. Works, since

$$\Delta_e(\mathcal{H}[C]) \leq \Delta_e(\mathcal{H}) \leq \left( \frac{b}{v(\mathcal{H})} \right)^{l-1} \frac{\varepsilon e(\mathcal{H})}{r'}$$

$$\leq \left( \frac{b}{v(\mathcal{H}[C])} \right)^{l-1} \cdot \frac{e(\mathcal{H}[C])}{r'}$$

"containers of containers"

NB: Independent sets of $\mathcal{H}$ stay independent

- # iterations at most $\frac{3}{5\varepsilon} \Rightarrow$ find $C$ has size

$$\exp(O_{\varepsilon}(n^{3/2} \log n))$$
Example triangle-free graphs: The container family

End result, phrased as graphs:

A collection $\mathcal{E}$ of $n$-vertex graphs and an $f : \mathcal{G}_n = \{ \text{graphs on } n \text{ vertices} \} \rightarrow \mathcal{E}$ s.t.
Example triangle-free graphs: The container family

End result, phrased as graphs:

A collection $\mathcal{E}$ of $n$-vertex graphs and an $f: G_n = \{\text{graphs on } n \text{ vertices}\} \rightarrow \mathcal{E}$ s.t.

- For every $\Delta$-free $n$-vertex $G$, ex. $S \in G_n$ with $e(S) \leq n^{3/2}$ and $S \subseteq G \subseteq f(S)$. 
Example triangle-free graphs: The container family

End result, phrased as graphs:

A collection $\mathcal{C}$ of $n$-vertex graphs and an
function $f : \mathcal{G}_n = \{\text{graphs on } n \text{ vertices}\} \rightarrow \mathcal{C}$ s.t.

- For every $\Delta$-free $n$-vertex $G$, each $S \subseteq \mathcal{G}_n$ with $e(S) \leq n^{3/2}$,
and $S \subseteq G \subseteq f(S)$.

- Every graph $C \in \mathcal{C}$ has less than $\Omega(n^2)$ triangles.
Example triangle-free graphs: The container family

End result, phrased as graphs:

A collection $\mathcal{C}$ of $n$-vertex graphs and an $f: \mathcal{G}_n = \{\text{graphs on } n \text{ vertices}\} \to \mathcal{C}$ s.t.

1. For every $\Delta$-free $n$-vertex $G$, ex. $S \in \mathcal{G}_n$ with $e(S) \leq n^{3/2}$ and $S \leq G \leq f(S)$.

2. Every graph $C \in \mathcal{C}$ has less than $\varepsilon \binom{n}{3} \approx \frac{\varepsilon}{6} n^3$ triangles.

3. $|\mathcal{C}| \leq \exp\left(c \cdot n^{3/2} \log n \right)$
Example triangle-free graphs: Counting

Goal: Show that there are \( 2^{n^2/4 + o(n^2)} \) \( \Delta \)-free graphs on \( n \) vertices.
Example triangle-free graphs: Counting

Goal: Show that there are $2^{n^2/4 + o(n^2)}$ Δ-free graphs on $n$ vertices.

Would be optimal-ish:

- the complete bipartite graph $K_{n/2,n/2}$ with $\frac{n^2}{4}$ edges, each of the $2^{n^2/4}$ subgraphs is Δ-free.
Example triangle-free graphs: Counting

Goal: Show that there are $2^{n^2/4 + o(n^2)}$ 1-free graphs on $n$ vertices.

Mantel (1907): Every triangle-free graph on $n$ vertices has $\leq \frac{n^2}{4}$ edges.

So $\Delta_n = \frac{n^2}{4}$ \Rightarrow we in fact show $2^{(1+o(1))\Delta_n}$
Example triangle-free graphs: Counting

Goal: Show that there are \(2^{n^2/4 + o(n^2)}\) triangle-free graphs on \(n\) vertices.

Mantel (1907): Every triangle-free graph on \(n\) vertices has \(\leq \frac{n^2}{4}\) edges.

Apply Mantel to small subgraphs to get supersaturation:
For every \(\delta > 0\) ex. \(\epsilon > 0\) s.t. any graph on \(\frac{n}{4} + \delta n^2\) edges contains \(\geq \epsilon n^3\) triangles.
Example triangle-free graphs: Counting

Goal: Show that there are \( 2^{n^2/4 + o(n^2)} \) \( \Delta \)-free graphs on \( n \) vertices.

Mantel (1907): Every triangle-free graph on \( n \) vertices has \( \leq \frac{n^2}{4} \) edges.

Apply Mantel to small subgraphs to get supersaturation:
For every \( \delta > 0 \) ex. \( \varepsilon > 0 \) s.t. any graph on \( \frac{n^2}{4} + \delta n^2 \) edges contains \( \geq \varepsilon n^3 \) triangles.

We combine this with our container family.
Example triangle-free graphs: Counting

For any \( \delta > 0 \), construct container family for the supercsturation \( \varepsilon(\delta) \)
Example triangle-free graphs: Counting

For any $\delta > 0$, construct container family for the supersaturation $\varepsilon(\delta)$

$\Rightarrow$ containers have $\leq 3n^3$ triangles

(by construction)
Example triangle-free graphs: Counting

For any $\delta > 0$, construct container family for the
supercsaturation $c(\delta)$

$\Rightarrow$ containers have $\leq 3n^3$ triangles

$\Rightarrow \leq \left( \frac{1}{\delta} + 8 \right)n^2$ edges

(by supercsaturation)
Example triangle-free graphs: Counting

For any $\delta > 0$, construct container family for the supercritical $\varepsilon(\delta)$

$n$ containers have $\leq \varepsilon n^3$ triangles

$\implies (1/n + \delta)n^2$ edges

$\implies$ The number of $\Delta$-free graphs on $n$ vertices is

$\leq \sum_{C \in \mathcal{C}} 2^{e(C)}$  (Every $\Delta$-free graph in some container)
**Example triangle-free graphs: Counting**

For any $\delta > 0$, construct container family for the superconcentration $\delta(n)$

$\Rightarrow$ containers have $\leq 3n^3$ triangles

$\Rightarrow < (\frac{1}{4} + \delta) n^2$ edges

$\Rightarrow$ The number of triangle-free graphs on $n$ vertices is

$$\leq \sum_{C \subseteq E} 2^{e(C)} \leq |E| \cdot 2^{(\frac{1}{4} + \delta) n^2} \leq \exp(8n^{3/2} \log n) \cdot 2^{(\frac{1}{4} + \delta) n^2}$$

by construction
Example triangle-free graphs: Counting

For any $\delta > 0$, construct a container family for the superseturation $\varepsilon(\delta)$

$\varepsilon(n^3)$ containers have $\leq \varepsilon n^3$ triangles

$\Rightarrow \leq (\frac{1}{4} + \delta) n^2$ edges

$\Rightarrow$ The number of $\Delta$-free graphs on $n$ vertices is

$$\leq \sum_{C \in \mathcal{C}} 2^{e(C)} \leq |E| \cdot 2^{(\frac{1}{4} + \delta)n^2} \leq \exp(\delta n^{3/2} \log n) \cdot 2^{(\frac{1}{4} + \delta)n^2}$$

$$= 2^{\frac{n^2}{4} + \delta n^2 + O(n^{3/2} \log n)}$$
Example triangle-free graphs: Typical structure

- Stability (Erdős-Simonovits/Füredi): \(\forall \delta > 0 \exists \varepsilon > 0\) s.t. any \(n\)-vertex graph with \(\geq \left(\frac{1}{2} - \varepsilon\right)\binom{n}{2}\) edges and \(<\varepsilon n^2\) 3's can be made bipartite by removing \(\leq \delta n^2\) edges.
Example triangle-free graphs: Typical structure

• Stability (Erdős-Simonovits/Füredi): ∀δ > 0 ∃ε > 0 s.t.
  any n-vertex graph with \( \geq (\frac{1}{2} - \varepsilon) \binom{n}{2} \) edges and \( < \varepsilon n^3 \) Δ's
  can be made bipartite by removing \( \leq \delta n^2 \) edges.

  "\( \delta n^2 \)-close to being bipartite"

Idea: Few triangles and edge number just under
  maximum for Δ-free \( \to \) close to extremal
  construction
Example triangle-free graphs: Typical structure

- Stability (Erdős-Simonovits/Füredi): \( \forall \delta > 0 \exists \epsilon > 0 \) s.t. any \( n \)-vertex graph with \( \geq \left( \frac{\epsilon}{2} \right) \binom{n}{2} \) edges and \( < \epsilon n^2 \) 3's can be made bipartite by removing \( \leq \delta n^2 \) edges.

- We will show: For every \( \alpha > 0 \) \( C > 0 \) s.t. almost all 3-free graphs with \( n \) vertices and \( m \geq Cn^{3/2} \log n \) edges is am-close to being bipartite.
Example triangle-free graphs: Typical structure

- **Stability (Erdős-Simonovits/Füredi):** \( \forall \delta > 0 \exists \varepsilon > 0 \) s.t. any \( n \)-vertex graph with \( \geq \left( \frac{1}{2} - \varepsilon \right) \binom{n}{2} \) edges and \( < \varepsilon n^2 \) \( \Delta \)'s can be made bipartite by removing \( \leq \delta n^2 \) edges.

- We will show: For every \( \alpha > 0 \exists \varepsilon > 0 \) s.t. almost all \( \Delta \)-free graphs with \( n \) vertices and \( m \geq C n^{3/2} \log n \) edges is \( \alpha m \)-close to being bipartite.

**Idea:** Count exceptions ("bad graphs") and compare to \( \binom{n^2/4}{n/2} \).

\[
\begin{align*}
\frac{n}{2} \leq \text{ any of the } \binom{n^2/4}{n/2} \text{ subgraphs is good!}
\end{align*}
\]
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are n-vertex graphs with $< 3n^2$ 3's? (After constructing them with 3 coming from stability w.r.t. some $\delta(d)$)
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are n-vertex graphs with $< 3n^3$ A's!

Stability $\Rightarrow$ either (a) $< \left( \frac{1}{2} - \epsilon \right) \binom{n}{2}$ edges

or (b) $\delta n^2$-close to bipartite (structure)
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are $n$-vertex graphs with $<3n^3$ $\Delta$'s!

Stability $\Rightarrow$ either (a) $<\left(\frac{1}{2}-\epsilon\right)n^2$ edges

or (b) $\delta n^2$-close to bipartite (structure)

Idea now: \* few edges $\Rightarrow$ few subgraphs with $m$ edges (good or bad)

\* containers have structure $\Rightarrow$ few graphs inside them without
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are $n$-vertex graphs with $<3n^3$ $\Delta$'s?

Stability $\Rightarrow$ either (a) $< (1/2 - \varepsilon)(n^3/2)$ edges

or (b) $\delta n^2$-close to bipartite (structure)

$\binom{n^2}{2} \leq (1 - \varepsilon)^m \binom{n^2/4}{m} \leq \exp\left(-\frac{\varepsilon}{2} m\right) \binom{n^2/4}{m}$

\[ \text{NB: This is for a single container of type } a) \]
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are \( n \)-vertex graphs with \(< 3n^3 \ \Delta's \)!

Stability \( \Rightarrow \) either (a) \(< (\frac{1}{2} - \varepsilon) \binom{n}{2} \) edges

\( \text{or (b) } \delta n^2 \text{-close to bipartite (structure)} \)

\[
(a) \quad \binom{\frac{1}{2} - \varepsilon}{m} \leq (1 - \frac{\varepsilon}{2})^m \binom{n^2/4}{m} \leq \exp\left(-\frac{\varepsilon}{2}m\right) \binom{n^2/4}{m}
\]

For (b): graph bad \( \Rightarrow \) \( > \delta m \) in the \( \leq \delta n^2 \) bad edges!

\( \text{(b) } \delta n^2 \binom{\frac{1}{2} + \delta}{(n^2 - \Delta m)} \binom{n^2}{(n-\Delta) m} \) \( \geq \) A "bad graph" needs to have \( \geq \delta m \) of its edges in the \( \leq \delta n^2 \) bad edges
Example triangle-free graphs: Typical structure

How to count bad graphs? Containers are $n$-vertex graphs with $\leq 3n^2$ 3's!

Stability $\Rightarrow$ either

(a) $\leq \binom{\frac{1}{2} - \epsilon}{2} n^2$ edges

or (b) $\delta n^2$-close to bipartite (structure)

\[
\binom{\left(\frac{1}{2} - \epsilon\right)n^2}{m} \leq \left(1 - \frac{\epsilon}{2}\right)^m \left(\frac{n^2}{4}\right) \leq \exp\left(-\frac{\epsilon}{2} m\right) \left(\frac{n^2}{4}\right)
\]

For (b): graph bad $\Rightarrow 2dm$ in the $\leq \delta n^2$ bad edges!

(b) $\frac{\delta n^2}{dm} \binom{\left(\frac{1}{4} + \delta\right)n^2}{(1 - \epsilon)n^2} \leq \left(\frac{\left(\frac{1}{4} + \delta\right)n^2}{m - dm}\right) \cdot \exp\left(-C_d m\right) \left(\frac{n^2}{4}\right) \left(\frac{\left(\frac{1}{4} + \delta\right)n^2}{m}\right) \leq \exp\left(-C_d m\right) \left(\frac{n^2}{4}\right)$
Example triangle-free graphs: Typical structure

So any single type (a) or (b) container results in $\leq \exp(-C_n m) \left( \frac{n^2}{4} \right)$ bad graphs.
Example triangle-free graphs: Typical structure

So any single type (a) or (b) container results in \( \leq \exp(-C_d m) \binom{n^2/4}{m} \) bad graphs

\[ \sim |E| = \exp(O(n^{3/2} \log n)) \rightarrow \text{can just union bound and still have } \leq \exp(-D(m)) \binom{n^2/4}{m} \text{ bad graphs!} \]
Example triangle-free graphs: Typical structure

So any single type (a) or (b) container results in \( \leq \exp(-C_n m) \binom{n^2/4}{m} \) bad graphs

\[ \sim |E| = \exp(O(n^{3/2} \log n)) \rightarrow \text{can just union bound and still have} \leq \exp(-\Delta(m)) \binom{n^2/4}{m} \text{ bad graphs!} \]

Negligible when compared to \( \binom{n^2/4}{m} \)!

LB on good graphs
Forbidden induced substructures

- Studying some questions for forbidden induced subgraphs also possible \( \Rightarrow \) vertices of \( \mathcal{E} \) will be pairs from \( E(K_n) \times \{0,1\} \) indicates whether we (don't) want an edge.
Forbidden induced substructures

- Studying same questions for forbidden induced subgraphs also possible \( \rightsquigarrow \) vertices of \( \mathcal{I} \) will be pairs from \( E(K_n) \times \{0, 1\} \)

- Problem: forgets structure (treats a graph \( F \) as a complete graph on \( \nu(F) \) vertices)
Forbidden induced substructures

• Studying same questions for forbidden induced subgraphs also possible \( \Rightarrow \) vertices of \( \mathcal{F} \) will be pairs from \( E(K_n) \times \{0, 1\} \)

• Problem: forgets structure

• Solution: Asymmetric (bipartite) container lemma (Morris-Samotij-Saxton)

Obtain typical structure of "induced-C_4-free" graphs \( \Rightarrow \) no \( \square \)
Forbidden induced substructures

- Studying same questions for forbidden induced subgraphs also possible \( \rightarrow \) vertices of \( \mathcal{H} \) will be pairs from \( E(K_n) \times \{0,1\} \)

- Problem: forgets structure

- Solution: Asymmetric (bipartite) container lemma (Morris-Samotij-Saxton)

- Generalized to multi-partite by Campos-Coulson-Serra-W.

How far can this be pushed?
THANK YOU!