The structure of Sidon set systems

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Sidon sets

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- Classical objects of study in additive number theory
- Idea: sets with little additive structure
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- **Idea:** sets with little additive structure

**Definition ($B_h$-set)**

For a positive integer $h$, a (finite) subset $A \subset G$ of an abelian group $G$ is called $B_h$-set if for any $a_1, \ldots, a_{2h} \in A$ it holds that

$$a_1 + \cdots + a_h = a_{h+1} + \cdots + a_{2h} \iff \{a_1, \ldots, a_h\} = \{a_{h+1}, \ldots, a_{2h}\}$$

as multisets.

A $B_2$-set is also called **Sidon set**.
This talk: interested in a generalization to set systems (families).

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Definition ($B_h$-system)

For a positive integer $h$, a family $\mathcal{F} \subset 2^G$ of subsets of an abelian group $G$ is called $B_h$-system if for any $A_1, \ldots, A_{2h} \in \mathcal{F}$ it holds that

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Note: Generalization, since a $B_h$-system consisting of singleton sets "is" exactly a $B_h$-set.
Previous results – bounds on Sidon systems

Today: Focus on the question: "How large can a Sidon set (system) be?"

Parametrization: $F_{k,h}(n)$ the largest size of a $B_h$-system in $\binom{[n]}{k}$, i.e. $k$-element subsets of $[n]$. For $h = 2$ (Sidon systems) we omit the $h$-subscript.
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Let $n \geq k \geq 2$ be integers. Then there exists a constant $C_k$ such that

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M. Wötz (UvA)
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- Also determined up to which size a *typical* family of \( k \)-subsets of \([n]\) will be a Sidon system.
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- Also determined up to which size a typical family of $k$-subsets of $[n]$ will be a Sidon system.
- **Remark:** $k = 1$ and $k \geq 2$ cases exhibit a gap ($\sqrt{n}$ vs $n^{k-1}$). Disappears if we allow $g \geq 2$ representations!
Main result

We (asymptotically) close the gap.

**Theorem**

Let $n, k, h \geq 2$ be integers. Then $F_{k,h}(n) = (1 \pm o(1))\binom{n}{k-1}$. 

Remarks: Also prove a result about largest $B_h$-system in binomial random subset of $\binom{n}{k}$. Both statements are straightforward consequences of a structural result.
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- Both statements are straight-forward consequences of a structural result.
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*For any positive integers $k$ and $h$, there exists an integer $\ell(k, h) = \ell$ such that the following holds.*
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$$A_1 + \cdots + A_h = B_1 + \cdots + B_h \iff \{A_1, \ldots, A_h\} = \{B_1, \ldots, B_h\},$$

where the equality on the right-hand side is as multisets.

Implies the first main result since there are only $O_{k,h}(n^{k-2})$ non-$B_\ell$-sets of cardinality $k$ in $[n]$ that contain e.g. 1, as long as $\ell$ only depends on $k$ and $h$. 
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**Lemma**

Let $A, B, D \subset G$ be subsets of an abelian group $G$ such that $A$ is a Sidon set. Then for any set $X \subset A$ satisfying $|X| > |B|$, it holds that

$$X + D \subset A + B \implies D \subset B.$$
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**Key insight** to use this lemma: If $A$ is a $B_{2\ell}$-set, then $A + X$ is a $B_\ell$-set for any $X \subset A$. 
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- **Toy example** \( A + B = A + D \) with \( k \geq 2 \): If \( A \) is \( B_4 \)-set, then \( A + A \) is a Sidon set and we have \( A + A + B = A + A + D \) and can apply the lemma.
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- **General case:** Instead of adding $A$ to both sides of $A + B = C + D$, we add $X_0 = A \cap C$. Can guarantee $|X_0| \geq 2$ and then repeat this with new intersection

$$X_1 = (X_0 + A) \cap (X_0 + C)$$

until at some point we are larger than $k$. 

Further outlook and open questions

- Key lemma holds in arbitrary abelian groups, structural theorem also in more general settings than \( \mathbb{Z} \) (e.g. \( \mathbb{R} \), groups admitting a total order)
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**Key point:** If we can guarantee \(|(A_i - A_i) \cap (B_j - B_j)| \geq 2\), then we can boost this to get full structural theorem for any abelian group.
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**Main open question:** Suppose \( G \) abelian group, \( A, B, C, D \subset G \) with same cardinality and \( A + B = C + D \). Does there always exist \( U \in \{A, B\}, V \in \{C, D\} \) that share a non-zero difference?

Some partial progress on this for specific groups and/or values of \( k \).

Other questions: Dependence of \( \ell \) on \( k \), sharp threshold in the constants \( c, C \) of the probabilistic statements, ...
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Thank you all for your attention!