EXERCISE CLASS 2-12-2016:
FMP AND DECIDABILITY, GENERAL FRAMES, PDL

(1) Recall that Den is the normal modal logic $K + (\Box\Box p \rightarrow \Box p)$
(a) Prove that the logic Den is sound and complete with respect to the class of dense Kripke frames.
(b) Show that the logic Den has the finite model property and is decidable.

(2) Let $\varphi$ be your favourite formula in the language of basic modal logic, not equivalent to falsum. Construct a general frame $f = (\mathcal{F}, A)$ such that $\mathcal{F} \not\models \varphi$ but $f \models \varphi$.

(3) Completeness with respect to general frames
(a) Let $\mathcal{M} = (\mathcal{F}, V)$ be Kripke model and let $A_{\mathcal{M}} := \{V(\varphi) : \varphi \in \text{Form}(\tau, \Phi)\}$. Show that $f_{\mathcal{M}} := (\mathcal{F}, A_{\mathcal{M}})$ is a general frame.
(b) Let $L$ be a consistent normal modal logic and let $f^L$ denote the general frame $f_{\mathcal{M}}$, where $\mathcal{M}^L$ is the canonical model for $L$. Show that $f^L \models L$.
(c) Conclude that any consistent normal modal logic is sound and (strongly) complete with respect to some class of general frames.

(4) Show that
(a) $\langle \pi_1, \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p$ is valid on a frame $(W, R_{\pi_1, \pi_2})$ iff $R_{\pi_1, \pi_2} = R_{\pi_1} \circ R_{\pi_2}$.
(b) $\langle \pi_1 \cup \pi_2 \rangle p \leftrightarrow \langle \pi_1 \rangle \langle \pi_2 \rangle p$ is valid on a frame $(W, R_{\pi_1, \pi_2})$ iff $R_{\pi_1 \cup \pi_2} = R_{\pi_1} \cup R_{\pi_2}$.
(c) $p \lor \langle \pi \rangle \langle \pi^* \rangle p \rightarrow \langle \pi^* \rangle p$ is valid on a frame $(W, R_{\pi, \pi^*})$ iff $(R_{\pi})^* \subseteq R_{\pi^*}$.

1. ADDITIONAL EXERCISE

(1) A logic $L$ is compact for the class $C$ of Kripke frames if the following condition is met: for every set of formulas $\Sigma$, if every finite subset of $\Sigma$ is satisfiable in a model based on a frame in $C$, then $\Sigma$ itself can be satisfied in a model based on a frame in $C$. Show that PDL is not compact for the class of regular frames.

(2) Prove the properties of atoms listed in Lemma 4.81.

(3) Prove that $At(\Sigma) = \{\Gamma \cap \neg FL(\Sigma) \mid \Gamma \text{ is MCS}\}$.

(4) Let $\Sigma = \{\varphi\}$ be a singleton set (of formulas in the language of PDL). Show that $\neg FL(\Sigma)$ is finite. Hint: This is not so easy.