(1) (30pt) Let $Z_i$ be a bisimulation between models $\mathcal{M}_1$ and $\mathcal{M}_2$, for each $i \in I$. Are the following relations bisimulations between $\mathcal{M}_1$ and $\mathcal{M}_2$:

- the union $\bigcup_{i \in I} Z_i$?
- the intersection $\bigcap_{i \in I} Z_i$?

Give a proof or a counter-example.

(2) (20pt) Let $\mathcal{M} = (\mathbb{N}, R, V)$ be a model such that $nRm$ iff $m = n + 1$, $V(p) = \{3k : k \in \mathbb{N}\}$ and $V(q) = \{3k + 2 : k \in \mathbb{N}\}$. Let also $\Sigma = \{p, q, \Diamond q\}$. As usual we assume that $0 \in \mathbb{N}$.

Describe the models $\mathcal{M}^s$ and $\mathcal{M}^l$, where $\mathcal{M}^s$ is the smallest and $\mathcal{M}^l$ is the largest filtration of $\mathcal{M}$ through $\Sigma$.

(3) (30pt) Prove that the filtrations $\mathcal{M}^s$ and $\mathcal{M}^l$ are indeed the smallest and largest filtrations, respectively.

(4) (20pt) Show that, given a transitive relation $R$, the relation $R^t$ (the transitive Lemmon filtration from Lemma 2.42 in the Blackburn, de Rijke, Venema book) is indeed a filtration and that any filtration of a transitive model that makes use of $R^t$ is guaranteed to be transitive.

(5) (BONUS!) (10pt) Show that any finite transitive filtration of a model based on the rationals with their usual ordering is a finite linear sequence of clusters, perhaps interspersed with singleton irreflexive points, no two of which can be adjacent.

Here a cluster on a transitive frame $(W, R)$ is a subset $C \subseteq W$ that is a maximal equivalence relation under $R$. That is, the restriction of $R$ to $C$ is an equivalence relation, and this is not the case for any other $D \subseteq W$ such that $C \subsetneq D$. 