MATHEMATICAL STRUCTURES IN LOGIC 2016
FINAL EXAM

- Deadline: April 5 — at 14:00.
- In exceptional cases the exam can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen `f.m.lauridsen@uva.nl`
- Grading is from 0 to 40 points.
- Success!

(1) (8pt) Let $\mathbf{LC} = \mathbf{IPC} + (\varphi \to \psi) \vee (\psi \to \varphi)$.

(a) Show that subdirectly irreducible **LC**-algebras (that is, HAs validating **LC**) are chains with a second greatest element.

(b) Show via $\to$-free reducts of HAs, that **LC** has the finite model property. That is, prove that if $\mathbf{LC} \nvdash \varphi$, then there is a finite **LC**-algebra A such that $A \not\models \varphi$. (Hint: use algebraic completeness and Birkhoff's theorem.)

(c) Characterize the lattice of subvarieties of the variety LC of all **LC**-algebras. (Hint: prove that every finite chain is a subalgebra of any infinite subdirectly irreducible **LC**-algebra.)

(2) (8pt) Let $(X, \leq)$ be a non-empty Priestley space.

(a) Show that the set of maximal points of $(X, \leq)$ is non-empty. (Hint: use Zorn's Lemma[1] and a version of compactness from Homework 4(4). In addition you can use the fact that for each $x \in X$ the set $\uparrow x$ is closed (Tutorial 5(4)).)

(b) Let $(X, \leq)$ be a Priestley space. Is the set of the maximal points $max(X)$ a closed set? Either provide a proof or give a counter-example.

(c) Give an example of a Stone space X and a partial order $\leq$ on X such that $\uparrow x$ is a closed set for each $x \in X$, but $(X, \leq)$ is not a Priestley space. (Hint: it might be useful to work with the two-point compactification of $\mathbb{N}$. That is, consider the space $\mathbb{N} \cup \{\infty_1, \infty_2\}$, whose topology is generated by the set

$$\mathcal{S} = \{\text{finite subsets of } \mathbb{N}, \text{cofinite subsets of } \mathbb{N} \text{ with } \{\infty_1, \infty_2\}, E \cup \{\infty_1\}, O \cup \{\infty_2\}\},$$

where E is the set of even numbers and O is the set of odd numbers. In other words we are taking the least topology containing $\mathcal{S}$).

[1]Recall that Zorn's Lemma states that if every chain in a nonempty poset has an upper bound, then the poset has a maximal element.

(3) (8pt) Let $\mathcal{B} = (B, \Box)$ be an $\mathsf{S4}$-algebra. A filter $F \subseteq B$ is called a *modal filter* if for each $a \in B$ we have

$$a \in F \;\Rightarrow\; \Box a \in F.$$

(a) Show that there is one-to-one correspondence between the congruences of $\mathcal{B}$ and modal filters of $\mathcal{B}$. (You can assume the correspondence between Boolean congruences and filters.)

(b) Let (X, R) be the $\mathsf{S4}$-space dual to $\mathcal{B}$. Characterize modal filters of $\mathcal{B}$ in dual terms. (You can assume the correspondence between filters of B and closed subsets of X.)

(c) Give a dual characterisation of subdirectly irreducible $\mathsf{S4}$-algebras. (Consult Homework 5(3) for a dual characterization of subdirectly irreducible Heyting algebras.)

(4) (8pt)

(a) Show that homomorphic images, subalgebras and products preserve the validity of equations. That is, if $A \models \varphi \approx \psi$ and B is a homomorphic image (or subalgebra) of A, then $B \models \varphi \approx \psi$, and if $A_i \models \varphi \approx \psi$ for each $i \in I$, and $B = \prod_{i \in I} A_i$, then $B \models \varphi \approx \psi$.

(b) Are all finitely generated Heyting algebras subdirectly irreducible? Give a proof or provide a counter example.

(c) Show that every finitely generated congruence distributive variety has only finitely many subvarieties. Recall that a vaiety V is finitely generated if there is a finite algebra A such that $\mathsf{V} = \mathbf{HSP}(A)$. (Hint: use Jónsson's lemma, 1.3 in the handout. You can assume that $\mathbf{P_U}(\mathsf{K}) = \mathsf{K}$ if K is a finite set of finite algebras. Here $\mathbf{P_U}$ stands for ultraproducts.)

(5) (8pt)

(a) Is there an intermediate logic having only a finite number of modal companions? Justify your answer.

(b) Let (X, R) be a finite quasi-ordered set. Show that for each $U \subseteq X$ we have $U \subseteq \Box_R(\Diamond_R(U))$ iff R is symmetric. Deduce that a finite $\mathsf{S4}$-algebra $\mathcal{B}$ is an $\mathsf{S5}$-algebra iff in its dual $\mathsf{S4}$-space the relation is an equivalence relation. (Recall that $\mathbf{S5} = \mathbf{S4} + (p \to \Box\Diamond p)$.)

(c) Is there a normal modal logic M with $\mathbf{S4} \subseteq M \subseteq \mathbf{S5}$ such that for no intermediate logic L we have $\tau(L) = M$? Justify your answer. (Hint: use finite algebras and duality. You can assume that $\mathbf{S5}$ is sound and complete wrt finite $\mathsf{S5}$-algebras.)