MATHEMATICAL STRUCTURES IN LOGIC 2016
HOMEWORK 2

- Deadline: February 16 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen (`f.m.lauridsen@uva.nl`)
- Grading is from 0 to 10 points.
- Success!

(1) (4pt) Do the following equations hold in any Heyting algebra? If yes, give a proof, if not, provide a counter-example.

(a) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$,

(b) $\neg\neg a \vee \neg a = 1$,

(c) $\neg\neg\neg a = \neg a$,

(d) $(a \to b) \vee (b \to a) = 1$.

Here $\neg a = a \to 0$.

(2) (2pt) Show that the lattice $(\mathrm{Fin}(\mathbb{N}) \cup \{\mathbb{N}\}, \subseteq)$ of finite subsets of $\mathbb{N}$ (together with $\mathbb{N}$) forms a complete bounded distributive lattice. Is this lattice a Heyting algebra?

(3) (2pt) Let L be a lattice. An element $a \in L$ is *compact* iff whenever $\bigvee A$ exists and $a \leq \bigvee A$ for $A \subseteq L$, then $a \leq \bigvee B$ for some finite $B \subseteq A$. We let $\mathbf{K}(L)$ denote the set of compact elements of L. For each of the following statements provide a proof or give a counterexample

(a) For each lattice L the set $\mathbf{K}(L)$ is closed under finite joins;

(b) For each lattice L the set $\mathbf{K}(L)$ is closed under finite meets.

(4) (2pt) We say that a lattice L is *compactly generated* if each element $a \in L$ is the supremum of a set of compact elements

(a) Show that every distributive lattice which is complete and compactly generated must necessarily be a Heyting algebra.

(b) Give an example of a complete Heyting algebra which is not compactly generated.