MATHEMATICAL STRUCTURES IN LOGIC 2016
HOMEWORK

- Deadline: March 1 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen `f.m.lauridsen@uva.nl`
- Grading is from 0 to 10 points.
- Success!

(1) (2pt) The aim of this exercise is to understand a duality of complete and atomic Boolean algebras and sets. This duality is closely related to Stone duality, but still differs from it.

A Boolean algebra B is called *atomic*, if given $a \neq 0$ in B, there exists an atom $b \in B$ such that $b \leq a$. Let **CABA** be the class of complete and atomic Boolean algebras. Let also **Set** be the class of all sets. To each set X we associate the powerset Boolean algebra $\mathcal{P}(X)$. To each complete and atomic Boolean algebra B we associate the set $\mathcal{A}(B)$ of its atoms. Show that

(a) Every complete and atomic Boolean algebra B is isomorphic to $\mathcal{P}(\mathcal{A}(B))$.

(b) Every set X is bijective to $\mathcal{A}(\mathcal{P}(X))$.

Categorical aspects of this correspondence will be discussed in the tutorial exercises.

(2) (2pt) Let L be a lattice and $A \subseteq L$ a non-empty set. Show that

$$[A) = \uparrow\{a_1 \wedge \cdots \wedge a_n : n \in \mathbb{N}, a_1, \ldots a_n \in A\}$$

is a filter, and moreover it is contained in any filter F of L which contains A.

(3) (3pt) A topological space X is called *extremally disconnected* if the closure of every open set in X is open (hence clopen since the closure is always closed). Prove that a Boolean algebra B is complete if and only if the Stone space X_B dual to B is extremally disconnected.

(4) (3pt) Let X be a Stone space. Prove that a map $\varepsilon : X \to X_{\mathsf{Clop}(X)}$ (where $X_{\mathsf{Clop}(X)}$ is the Stone space dual to $\mathsf{Clop}(X)$) defined by $\varepsilon(x) = \{U \in \mathsf{Clop}(X) : x \in U\}$ is a well-defined bijection and that for each clopen in $X_{\mathsf{Clop}(X)}$ its ε-pre-image is clopen in X.

The following characterization of compactness might be useful: a space X is compact if and only if for any family $\mathcal{C}$ of closed sets with the finite intersection property we have $\bigcap \mathcal{C} \neq \emptyset$. Note that $\mathcal{C} = \{C_i : i \in I\}$ has the finite intersection property iff for any finite $J \subseteq I$ the intersection $\bigcap\{C_i : i \in J\} \neq \emptyset$.