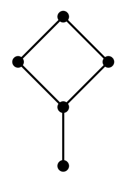
MATHEMATICAL STRUCTURES IN LOGIC 2016
HOMEWORK 6

- Deadline: March 15 — at the **beginning** of class.
- In exceptional cases homework can be submitted electronically (in a single pdf-file!) to Frederik Lauridsen (`f.m.lauridsen@uva.nl`)
- Grading is from 0 to 10 points.
- Success!

(1) (2pt) Find a subdirect representation of the following Heyting algebra A



That is, find an embedding $\iota\colon A \hookrightarrow \prod_{i \in I} A_i$ of HAs such that for each $i \in I$ the algebra A_i is subdirectly irreducible and $\pi_i \circ \iota\colon A \to A_i$ is surjective.

(2) (2pt) Prove that for Heyting algebras A and B we have $A \times B \simeq \mathsf{ClpUp}(X_A \sqcup X_B)$, where $X_A \sqcup X_B$ is the disjoint union of X_A and X_B

(3) (4pt) (Finite model property)
  (a) Give full details of the algebraic proof[1] of the finite model property of HAs (**IPC**).
  (b) Show that if a finite HA $A \not\models \varphi$, then there is a subdirectly irreducible algebra $B \in \mathbf{H}(A)$ such that $B \not\models \varphi$. (You may use duality.)

  Conclude that the variety of all Heyting algebras is generated by its finite subdirectly irreducible members.

(4) (2pt)
  (a) Find Priestley spaces $\mathcal{X} = (X, \leq)$ and $\mathcal{Y} = (Y, \leq)$ and a continuous order-preserving map $f : X \to Y$ such that $f^{-1} : \mathsf{ClpUp}(\mathcal{Y}) \to \mathsf{ClpUp}(\mathcal{X})$ does not preserve Heyting implication.

  (b) Show that $f\colon X \to Y$ is a continuous p-morphism[2] between Esakia spaces $\mathcal{X} = (X, \leq)$ and $\mathcal{Y} = (Y, \leq)$ iff $f^{-1}\colon \mathsf{ClpUp}(\mathcal{Y}) \to \mathsf{ClpUp}(\mathcal{X})$ is a Heyting algebra homomorphism.

[1] A sketch of this proof will be given in Thursday's class.

[2] Continuous p-morphism are also called *Esakia morphisms*.

Recall that a map $f : P \to Q$ between posets is called a *p-morphism* (or *bounded morphisms*) iff it is order preserving and satisfies

$$\forall a \in P \, \forall b \in Q \, (f(a) \leq b \implies \exists a' \in P \, (a \leq a' \,\&\, f(a') = b)).$$