Varieties of Heyting algebras and superintuitionistic logics

Nick Bezhanishvili
Institute for Logic, Language and Computation
University of Amsterdam
http://www.phil.uu.nl/~bezhanishvili
email: N.Bezhanishvili@uva.nl

Heyting algebras

A Heyting algebra is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ equipped with a binary operation $\rightarrow$, which is a right adjoint of $\wedge$. This means that for each $a, b, x \in A$ we have

$$a \wedge x \leq b \text{ iff } x \leq a \rightarrow b.$$

Heyting algebras

Heyting algebras pop up in different areas of mathematics.

1. **Logic**: Heyting algebras are algebraic models of intuitionistic logic.
2. **Topology**: opens of any topological space form a Heyting algebra.
3. **Geometry**: open subpolyhydra of any polyhedron form a Heyting algebra.
4. **Category theory**: subobject classifier of any topos is a Heyting algebra.
5. **Universal algebra**: lattice of all congruences of any lattice is a Heyting algebra.

Outline

The goal of the tutorial is to give an insight into the complicated structure of the lattice of varieties of Heyting algebras.

The outline of the tutorial:

1. Heyting algebras and superintuitionistic logics
2. Representation of Heyting algebras
3. Hosoi classification of the lattice of varieties of Heyting algebras
4. Jankov formulas and splittings
5. Canonical formulas

Part 1: Heyting algebras and superintuitionistic logics

Constructive reasoning

One of the cornerstones of classical reasoning is the law of excluded middle $p \vee \neg p$.

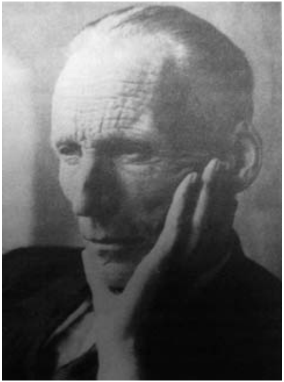
Constructive viewpoint: Truth = Proof.

The law of excluded middle $p \vee \neg p$ is constructively unacceptable.

For example, we do not have a proof of Goldbach's conjecture nor are we able to show that this conjecture does not hold.

Constructive reasoning

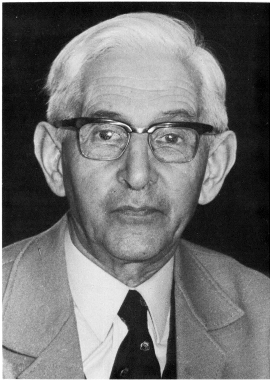
On the grounds that the only accepted reasoning should be constructive, the dutch mathematician L. E. J. Brouwer rejected classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

Intuitionistic logic

In 1930's Brouwer's ideas led his student Heyting to introduce intuitionistic logic which formalizes constructive reasoning.



Arend Heyting (1898 - 1980)

Intuitionistic logic

Roughly speaking, the axiomatization of intuitionistic logic is obtained by dropping the law of excluded middle from the axiomatization of classical logic.

CPC = classical propositional calculus
IPC = intuitionistic propositional calculus.

The law of excluded middle is not derivable in intuitionistic logic. So **IPC** $\subsetneq$ **CPC**.

In fact,

$$\mathbf{CPC} = \mathbf{IPC} + (p \vee \neg p).$$

There are many logics in between **IPC** and **CPC**

Superintuitionistic logics

A superintuitionistic logic is a set of formulas containing **IPC** and closed under the rules of substitution and Modus Ponens.

Superintuitionistic logics contained in **CPC** are often called intermediate logics because they are situated between **IPC** and **CPC**.

As we will see, intermediate logics are exactly the consistent superintuitionistic logics.

Since we are interested in consistent logics, we will mostly concentrate on intermediate logics.

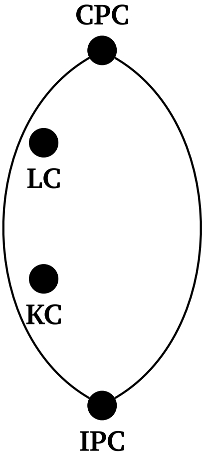
Intermediate logics

$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Equational theories of Heyting algebras

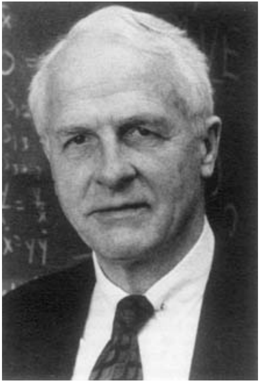
Each formula φ in the language of **IPC** corresponds to an equation $\varphi \approx 1$ in the theory of Heyting algebras.

Conversely, each equation $\varphi \approx \psi$ can be rewritten as $\varphi \leftrightarrow \psi \approx 1$, which corresponds to the formula $\varphi \leftrightarrow \psi$.

This yields a one-to-one correspondence between superintuitionistic logics and equational theories of Heyting algebras.

Varieties of Heyting algebras

By the celebrated Birkhoff theorem, equational theories correspond to varieties; that is, classes of algebras closed under subalgebras, homomorphic images, and products.



Garrett Birkhoff (1911 - 1996)

Varieties of Heyting algebras

Thus, superintuitionistic logics correspond to varieties of Heyting algebras, while intermediate logics to non-trivial varieties of Heyting algebras.

Heyt = the variety of all Heyting algebras.

Bool = the variety of all Boolean algebras.

$\Lambda(\mathbf{IPC})$ = the lattice of superintuitionistic logics.

$\Lambda(\mathbf{Heyt})$ = the lattice of varieties of Heyting algebras.

Theorem. $\Lambda(\mathbf{IPC})$ is dually isomorphic to $\Lambda(\mathbf{Heyt})$.

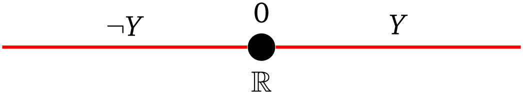
Consequently, we can investigate superintuitionistic logics by means of their corresponding varieties of Heyting algebras.

Part 2: Representation of Heyting algebras

First typical example of a Heyting algebra

Open sets of any topological space X form a Heyting algebra, where for open $Y, Z \subseteq X$:

$$Y \to Z = \mathrm{Int}(Y^c \cup Z), \quad \neg Y = \mathrm{Int}(Y^c).$$



$$Y \vee \neg Y \neq \mathbb{R}$$

Stone Representation

Theorem (Stone, 1937). Every Heyting algebra can be embedded into the Heyting algebra of open sets of some topological space.

Marshall Stone (1903 - 1989)

Stone representation

For every Heyting algebra A let X_A be the set of prime filters of A.

The Stone map $\varphi : A \to \mathcal{P}(X_A)$ is given by

$$\varphi(a) = \{x \in X_A : a \in x\}.$$

Let Ω_A be the topology generated by the basis $\{\varphi(a) : a \in A\}$.

Theorem. $\varphi : A \to \Omega_A$ is a Heyting algebra embedding.

Second typical example of a Heyting algebra

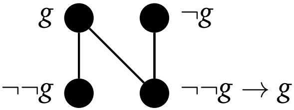
Up-sets of any poset $(X, \leq)$ form a Heyting algebra where for up-sets $U, V \subseteq X$:

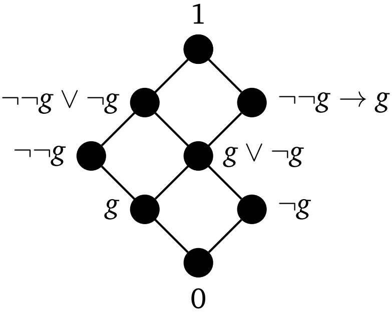
$$U \to V = X - {\downarrow}(U - V), \quad \neg U = X - {\downarrow}U$$

Here U is an up-set if $x \in U$ and $x \leq y$ imply $y \in U$ and

${\downarrow}U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.

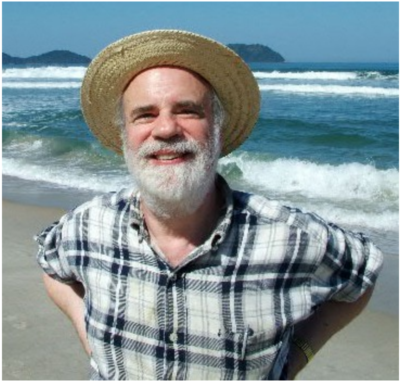
Second typical example of a Heyting algebra





Kripke Representation

Theorem (Kripke, 1965). Every Heyting algebra can be embedded into the Heyting algebra of up-sets of some poset.



Saul Kripke

Kripke representation

For every Heyting algebra A, order the set X_A of prime filters of A by set-theoretic inclusion.

For a poset X let $\mathrm{Up}(X)$ be the Heyting algebra of up-sets of X.

Theorem. The Stone map $\varphi : A \to \mathrm{Up}(X_A)$ is a Heyting algebra embedding.

We want to characterize the φ-image of A.

For this we will define a topology on X_A and characterize this image in order-topological terms.

This topology will be the so-called patch topology of Ω_A.

Esakia duality

This approach was developed by Esakia in the 1970's.

Leo Esakia (1934 - 2010)

Esakia duality

An Esakia space is a pair $(X, \leq)$, where:

1. X is a Stone space (compact, Hausdorff, zero-dimensional).
2. $(X, \leq)$ is a poset.
3. ${\uparrow}x$ is closed for each $x \in X$. Here ${\uparrow}x = \{y \in X : x \leq y\}$.
4. If U is clopen (closed and open), then so is ${\downarrow}U$. Recall that ${\downarrow}U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.

Esakia duality

Given an Esakia space $(X, \leq)$ we take the Heyting algebra $(\mathrm{CpUp}(X), \cap, \cup, \rightarrow, \emptyset, X)$ of all clopen up-sets of X, where for $U, V \in \mathrm{CpUp}(X)$:

$$U \rightarrow V = X - {\downarrow}(U - V).$$

For each Heyting algebra A we take the set X_A of prime filters of A ordered by inclusion and topologized by the subbasis

$$\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}.$$

Alternatively we can take $\{\varphi(a) - \varphi(b) : a, b \in A\}$ as a basis for the topology.

Esakia Duality

Theorem.

1. For each Heyting algebra A the map $\varphi : A \to \mathrm{CpUp}(X_A)$ is a Heyting algebra isomorphism.
2. For each Esakia space X, there is an order-hemeomorphism between X and $X_{\mathrm{CpUp}(X)}$.

This is the object part of the duality between the category of Heyting algebras and Heyting algebra homomorphisms and the category of Esakia spaces and Esakia morphisms.

Priestley spaces

Order-topological representation of bounded distributive lattices was developed by Priestley in the 1970s.

Hilary Priestley

Priestley spaces

In each Esakia space the following Priestley separation holds:

$x \not\leq y$ implies there is a clopen up-set U such that $x \in U$ and $y \notin U$.

Thus, every Esakia space is a Priestley space, but not vice versa.

It follows that Esakia duality is a restricted version of Priestley duality.

Recap

1. The lattice of superintuitionistic logics is dually isomorphic to the lattice of varieties of Heyting algebras.
2. Stone Representation: Every Heyting algebra can be embedded into the Heyting algebra of open sets of some topological space.
3. Kripke Representation: Every Heyting algebra can be embedded into the Heyting algebra of up-sets of some poset.
4. Esakia Representation: Every Heyting algebra is isomorphic to the Heyting algebra of clopen up-sets of some Esakia space.

Part 3: Depth and Hosoi classification

Depth of Heyting algebras

Let $(X, \leq)$ be a poset.

1. We say that X is of depth $n > 0$, denoted $d(X) = n$, if there is a chain of n points in X and no other chain in X contains more than n points. The poset X is of finite depth if $d(X) = n$ for some $n > 0$.

2. We say that X is of infinite depth, denoted $d(X) = \omega$, if for every $n \in \omega$, X contains a chain consisting of n points.

Depth is also referred to as height.

Let A be a Heyting algebra.

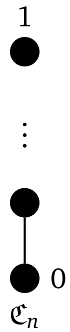
The depth $d(A)$ of A = the depth of the dual of A.

Let $\mathbf{V}$ be a variety of Heyting algebras.

The depth $d(\mathbf{V})$ of $\mathbf{V}$ = $\sup\{d(A) : A \in V\}$.

Chains

Let $\mathfrak{C}_n$ be the n-element chain.



$\mathfrak{C}_n$ is a Heyting algebra, where

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

$$d(\mathfrak{C}_{n+1}) = n$$

Varieties of depth n

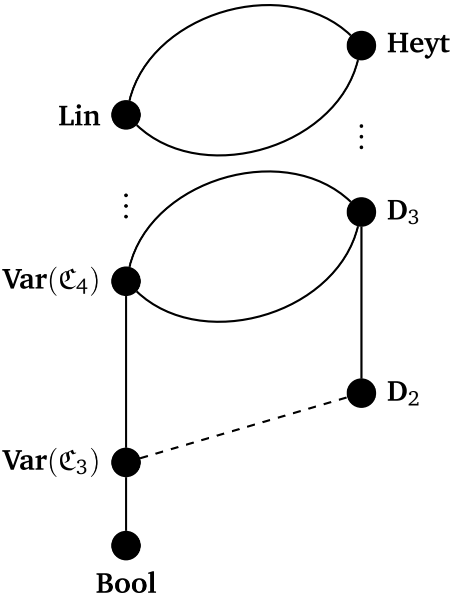
For a class K of Heyting algebras, let $\mathbf{Var}(\mathsf{K})$ be the variety of Heyting algebras generated by K.

Let **Lin** be the variety generated by all finite chains.

Let also $\mathbf{D}_n$ be the class of all Heyting algebras of depth n.

We will see later that each $\mathbf{D}_n$ forms a variety.

Rough picture of the lattice



Part 4: Jankov formulas and the cardinality of the lattice of varieties

Filters and congruences

As in Boolean algebras, the lattice of filters of a Heyting algebra is isomorphic to the lattice of congruences.

To each filter F corresponds the congruence θ_F defined by

$$a\theta_F b \text{ if } a \leftrightarrow b \in F.$$

To each congruence θ corresponds the filter

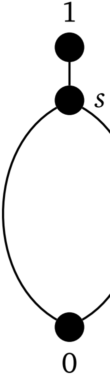
$$F_\theta = \{a \in A : a\theta 1\}.$$

Consequently, the variety of Heyting algebras is congruence distributive and has the congruence extension property.

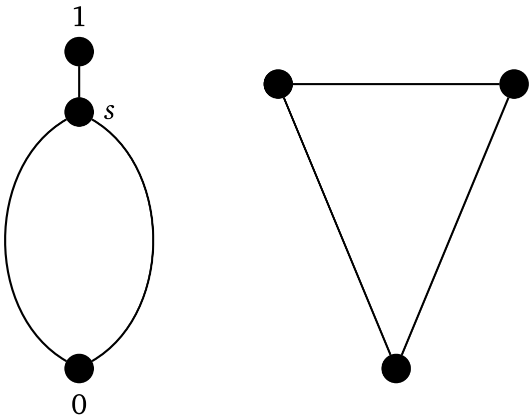
Subdirectly irreducible Heyting algebras

By another theorem of Birkhoff, every variety of algebras is generated by its subdirectly irreducible members.

Theorem (Jankov, 1963). A Heyting algebra is subdirectly irreducible (s.i. for short) if it has a second largest element.



Esakia duals of s.i. Heyting algebras



If a Heyting algebra A is s.i., then the dual of A has a least element, a root.

If an Esakia space is rooted and the root is an isolated point, then its dual Heyting algebra is s.i.

Jankov formulas

Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A.

For each $a \in A$ we introduce a new variable p_a and define the Jankov formula $\chi(A)$ as the $(\wedge, \vee, \to, 0, 1)$-description of this algebra.

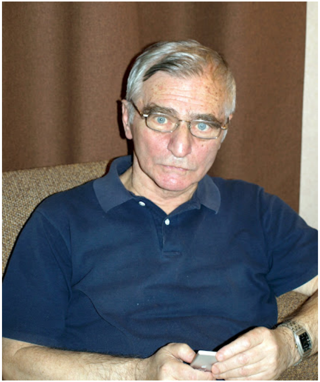
$$\begin{array}{rcl} \chi(A) & = & [\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ & & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\wedge \\ & & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\}\wedge \\ & & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}] \to p_s \end{array}$$

If we interpret p_a as a, then the Jankov formula of A is equal in A to s, i.e., it is pre-true in A .

Axiomatization of varieties of Heyting algebras

Theorem (Jankov, 1963). Let B a Heyting algebra. Then

$$B \not\models \chi(A) \text{ iff } A \in \mathbf{SH}(B).$$



Dimitri Jankov

Axiomatization of varieties of Heyting algebras

Theorem (Jankov, 1963). Let B a Heyting algebra. Then

$$B \not\models \chi(A) \text{ iff } A \in \mathbf{SH}(B).$$

Proof. (Sketch). Suppose $B \not\models \chi(A)$. Then there exists a s.i. homomorphic image C of B such that $C \not\models \chi(A)$. Moreover $\chi(A)$ is pre-true in C. This means that there is a valuation v on C such that

$$\begin{aligned}
&v(\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\
&\bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\}\wedge \\
&\bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\}\wedge \\
&\bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}) = 1_C
\end{aligned}$$

and

$$v(p_s) = s_C$$

Axiomatization of varieties of Heyting algebras

Therefore, for all $a, b \in A$ we have:

$$\begin{aligned} v(p_{a \wedge b}) &= v(p_a) \wedge v(p_b) \\ v(p_{a \vee b}) &= v(p_a) \vee v(p_b) \\ v(p_{a \to b}) &= v(p_a) \to v(p_b) \\ v(p_{\neg a}) &= \neg v(p_a) \\ v(p_s) &= s_C \end{aligned}$$

We consider the map $h : A \to C$ given by $h(a) = v(p_a)$.

Then h is a Heyting embedding.

Conversely, as $A \not\models \chi(A)$ and $A \in \mathbf{SH}(B)$ we see that $B \not\models \chi(A)$.

Splittings

Jankov formulas are used to axiomatize many varieties of Heyting algebras.

For example, they axiomatize all splitting varieties of Heyting algebras.

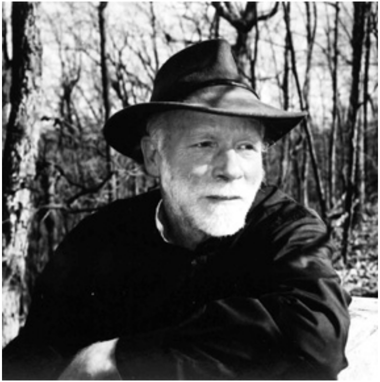
Splittings started to play an important role in lattice theory in the 1940s.

A pair (a, b) splits a lattice L if $a \not\leq b$ and for each $c \in L$:

$$a \leq c \text{ or } c \leq b$$

Splittings

R. McKenzie in the 1970’s revisited splittings when he started an extensive study of lattices of varieties.



Ralph McKenzie

Splittings

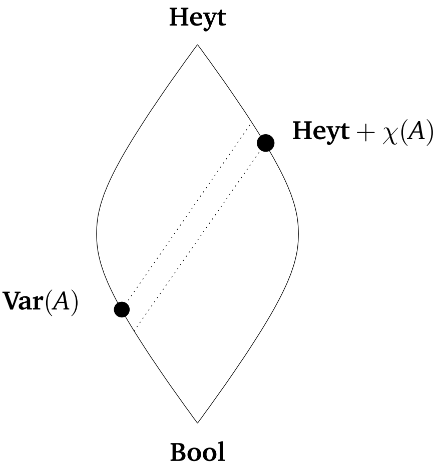


Figure: Splitting of the lattice of varieties of Heyting algebras

Splittings

Theorem. For each subdirectly irreducible Heyting algebra A the pair $(\mathbf{Var}(A), \mathbf{Heyt} + \chi(A))$ splits the lattice of varieties of Heyting algebras.

Proof. (Sketch) Since $A \not\models \chi(A)$, we see that $\mathbf{Var}(A) \not\subseteq \mathbf{Heyt} + \chi(A)$.

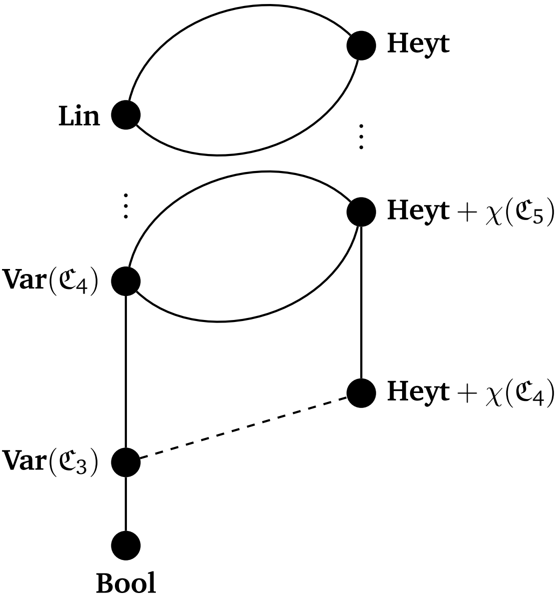
Suppose $\mathbf{V}$ is a variety such that $\mathbf{V} \not\subseteq \mathbf{Heyt} + \chi(A)$.

Then there is $B \in \mathbf{V}$ such that $B \not\models \chi(A)$.

By Jankov's theorem, $A \in \mathbf{SH}(B)$ and so $\mathbf{Var}(A) \subseteq \mathbf{V}$.

The other direction follows from a result of McKenzie (1972).

Rough picture of the lattice



Continuum of varieties of Heyting algebras

Let A and B be s.i. Heyting algebras. We write $A \leq B$ if $A \in \mathbf{SH}(B)$.

Theorem. If Δ is an $\leq$-antichain of finite s.i. algebras, then for each $I, J \subseteq \Delta$ with $I \neq J$, we have

$$\mathbf{Heyt} + \{\chi(A) : A \in I\} \neq \mathbf{Heyt} + \{\chi(A) : A \in J\}.$$

Proof. (Sketch) If $I \not\subseteq J$, then there is $B \in I$ such that $B \notin J$.

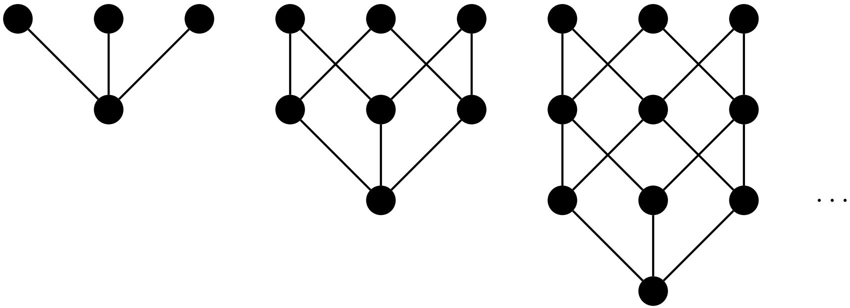
Then $A \not\leq B$ for each $A \in J$. Therefore, by Jankov's theorem, $B \models \chi(A)$ for each $A \in J$.

So $B \in \mathbf{Heyt} + \{\chi(A) : A \in J\}$.

But $B \not\models \chi(B)$. So $B \notin \mathbf{Heyt} + \{\chi(A) : A \in I\}$.

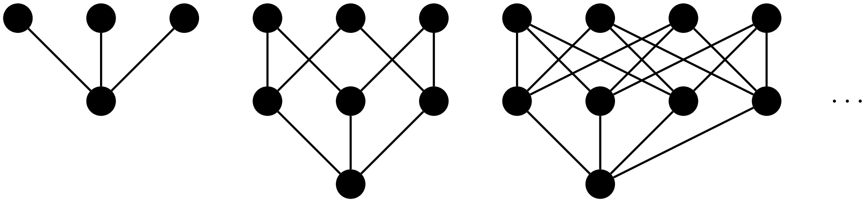
How can we construct an $\leq$-antichain of finite s.i. algebras?

Antichains



Lemma. Δ_1 is an $\leq$-antichain.

Antichains



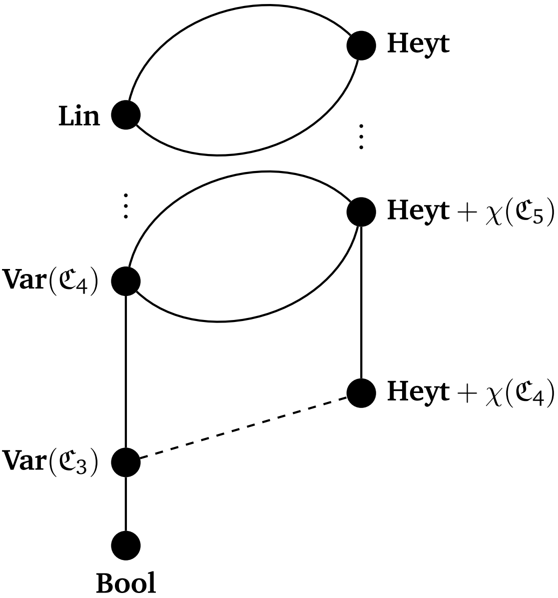
Lemma. Δ_2 is an $\leq$-antichain.

Continuum of varieties of Heyting algebras

Corollary.

1. There is a continuum of varieties of Heyting algebras.
2. In fact, there is a continuum of varieties of Heyting algebras of depth 3.
3. And there is a continuum of varieties of Heyting algebras of width 3.

Rough picture of the lattice



Varieties axiomatized by Jankov formulas

Is every variety of Heyting algebras axiomatized by Jankov formulas?

A variety **V** is locally finite if every finitely generated **V**-algebra is finite.

Theorem Every locally finite variety of Heyting algebras is axiomatized by Jankov formulas.

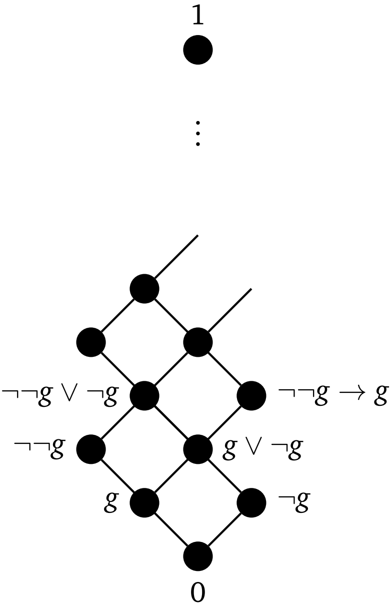
Corollary. Varieties of finite depth are locally finite and hence axiomatized by Jankov formulas.

Finitely generated algebras

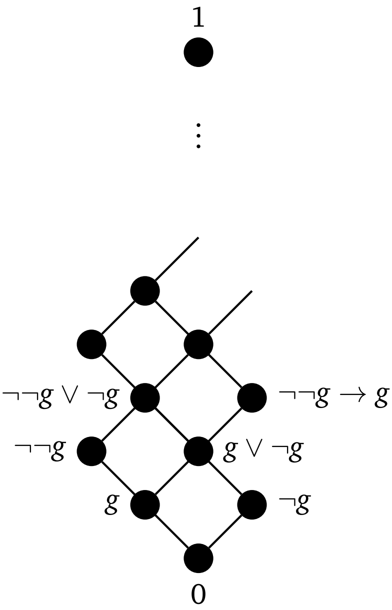
However, there are continuum many non-locally finite varieties of Heyting algebras.

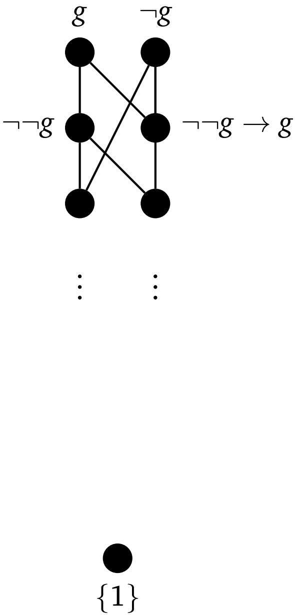
Theorem (Rieger, 1949, Nishimura, 1960). The 1-generated free Heyting algebra, also called the Rieger-Nishimura lattice, is infinite.

The Rieger-Nishimura Lattice



1-generated free Heyting algebra





Axiomatization of varieties of Heyting algebras

There exist varieties of Heyting algebras that are not axiomatized by Jankov formulas.

Problem: Can we generalize Jankov's method to all varieties of Heyting algebras?

Recap

1. Classification of the lattice of varieties of Heyting algebras via their depth.
2. Subdirectly irreducible Heyting algebras and their dual Esakia spaces.
3. Jankov formulas and splitting varieties.
4. Continuum of varieties of Heyting algebras via Jankov formulas.
5. **Problem**: Can we generalize Jankov's method to all varieties of Heyting algebras?

Part 5: Canonical formulas

Axiomatization of varieties of Heyting algebras

The affirmative answer was given by Michael Zakharyaschev via canonical formulas.

Michael Zakharyaschev

Locally finite reducts

We will give an algebraic account of this method.

Although Heyting algebras are not locally finite, they have locally finite reducts.

Heyting algebras $(A, \wedge, \vee, \to, 0, 1)$.

$\vee$-free reducts $(A, \wedge, \to, 0, 1)$: implicative semilattices.

$\to$-free reducts $(A, \wedge, \vee, 0, 1)$: distributive lattices.

Theorem.

- (Diego, 1966). The variety of implicative semilattices is locally finite.
- (Folklore). The variety of distributive lattices is locally finite.

$(\wedge, \to)$-canonical formulas

We will use these reducts to derive desired axiomatizations of varieties of Heyting algebras.

First we will need to extend the theory of Jankov formulas.

Jankov formulas describe the full Heyting signature. We will now look at $\vee$-free reducts.

The homomorphisms will now preserve only $\wedge$, 0 and $\to$. In general they do not preserve $\vee$. But they may preserve some joins.

This can be encoded in the following formula.

$(\wedge, \to)$-canonical formulas

Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2.

For each $a \in A$ we introduce a new variable p_a and define the $(\wedge, \to)$-canonical formula $\alpha(A, D)$ associated with A and D as

$$\begin{aligned}\alpha(A,D) \quad = \quad & [\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\}\wedge \\ & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\wedge \\ & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}] \to p_s\end{aligned}$$

Note that if $D = A^2$, then $\alpha(A, D) = \chi(A)$.

$(\wedge, \to)$-canonical formulas

Theorem. Let A be a finite s.i. Heyting algebra, $D \subseteq A^2$, and B a Heyting algebra. Then

$B \not\models \alpha(A, D)$ iff there is a homomorphic image C of B and an $(\wedge, \to)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$.

Theorem. Every variety of Heyting algebras is axiomatized by $(\wedge, \to, 0)$-canonical formulas.

We show that for each formula φ there exist finitely many $A_1, \ldots, A_m$ and $D_i \subseteq A_i^2$ such that

$$\mathbf{Heyt} + \varphi = \mathbf{Heyt} + \alpha(A_1, D_1) + \cdots + \alpha(A_m, D_m)$$

$(\wedge, \to)$-canonical formulas

Proof idea. Suppose $B \not\models \varphi$.

Then there exist elements $a_1, \ldots, a_n \in B$ on which φ is refuted.

We generate the implicative semilattice $(A, \wedge, \to, 0)$ of B by the subpolynomials Σ of $\varphi(a_1, \ldots, a_n)$.

By Diego's theorem $(A, \wedge, \to, 0)$ is finite.

$(\wedge, \to)$-canonical formulas

We define a "fake" $\dot{\vee}$ on A by $a \dot{\vee} b = \bigwedge\{s \in A : s \geq a, b\}$. Then $(A, \wedge, \dot{\vee}, 0, \to)$ is a finite Heyting algebra. Also for $a, b \in A$ we have

$$a \vee b \leq a \dot{\vee} b.$$

Moreover, if $a \vee b \in \Sigma$ then

$$a \vee b = a \dot{\vee} b.$$

This implies that the algebra $(A, \wedge, \dot{\vee}, \to, 0)$ refutes φ.

$(\wedge, \to)$-canonical formulas

Now we let $D = \{(a, b) : a \vee b \in \Sigma\}$.

Then

$$A \overset{i}{\hookrightarrow} B$$

i is a $(\wedge, \to, 0)$-embedding, preserving $\vee$ on the elements of D.

A may not be s.i.

$(\wedge, \to)$-canonical formulas

We take a s.i. homomorphic image A' of A (such can always be found) via some κ that refutes φ. We also let D' be the κ-image of D. So

$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & B \\ \big\downarrow{\scriptstyle \kappa} & & \\ A' & & \end{array}$$

i is a $(\wedge, \to, 0)$-embedding, preserving $\vee$ on the elements of D, and κ is a Heyting homomorphism.

$(\wedge, \to)$-canonical formulas

Implicative semilattices have the congruence extension property. Thus, there is an implicative semillatice C such that

$$\begin{array}{ccc} A & \overset{i}{\hookrightarrow} & B \\ \kappa \downarrow & & \downarrow \xi \\ A' & \overset{h}{\hookrightarrow} & C \end{array}$$

Onto $(\wedge, \to, 0)$-homomorphisms are Heyting homomorphisms, so C is a Heyting algebra that is a homomorphic image of B.

Moreover, h preserves $\vee$ on the elements of D'.

So we found a finite s.i. algebra A' and a set $D' \subseteq A'^2$ such that A' is $(\wedge, \to, 0)$-embedded into a homomorphic image of B preserving $\vee$ on D'.

$(\wedge, \to)$-canonical formulas

So B refutes $\alpha(A', D')$.

Let $k = |Sub(\varphi)|$.

By Diego's theorem there is $M(k)$ such that every k-generated implicative semilattice has less than $M(k)$-elements.

Let $A_1, \ldots, A_m$ be the list of all (finitely many) Heyting algebras of size $M(k)$-refuting φ.

Let V_i be a valuation refuting φ in A_i. Set

$$\Sigma_i = \{V_i(\psi) : \psi \in Sub(\varphi)\}.$$

Let $D_i = \{(a, b) : a \vee b \in \Sigma_i\}$.

By construction $|A'| < M(k)$. So $(A', D') = (A_i, D_i)$ for some $i \leq m$.

$(\wedge, \to)$-canonical formulas

Thus, we proved that $B \not\models \varphi$ implies $B \not\models \alpha(A_i, D_i)$ for some $i \leq m$.

Conversely, let $B \not\models \alpha(A_i, D_i)$ for some $i \leq m$.

Then here is a homomorphic image C of B and an $(\wedge, \to, 0)$-embedding $h : A_i \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D_i$.

By construction of D_i we have that $C \not\models \varphi$.

So $B \not\models \varphi$.

Thus, we proved

$$\mathbf{Heyt} + \varphi = \mathbf{Heyt} + \alpha(A_1, D_1) + \cdots + \alpha(A_m, D_m)$$

Therefore, every variety of Heyting algebras is axiomatized by $(\wedge, \to)$-canonical formulas.

Subframe formulas

$\alpha(A, A^2) = \chi(A)$.

$\alpha(A, \emptyset)$ is called a subframe formula.

Subframes play the same role here as submodels in model theory.

Theorem. Let A be a finite s.i. algebra and X_A its dual space. A Heyting algebra B refutes $\alpha(A)$ iff X_A is a subframe X_B.

$(\wedge, \rightarrow)$-embeddability means that we take subframes of the dual space.

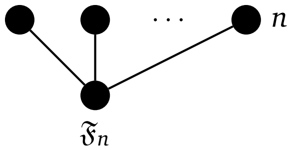
There are continuum many logics axiomatized by such formulas.

All subframe logics have the finite model property.

Subframe formulas

Theorem: Let A be a s.i. Heyting algebra and X_A its dual space. Then

- X_A has width $< n$ iff n-fork is not a subframe of X_A iff $\alpha(\mathfrak{F}_n)$ is true in A.



A variety of Heyting algebras **V** is of width $< n$ if the width of X_A is $< n$ for each s.i. $A \in \mathbf{V}$

$$\mathbf{V} \text{ is of width } < n \text{ iff } A \models \alpha(\mathfrak{F}_n), \text{ for each } A \in \mathbf{V}$$

$(\wedge, \vee)$-canonical formulas

We can also develop the theory of $(\wedge, \vee)$-canonical formulas $\gamma(A, D)$ using the $\to$-free locally finite reducts of Heyting algebras.

The theory of these formulas is different than that of $(\wedge, \to)$-canonical formulas.

Theorem. Every variety of Heyting algebras is axiomatized by $(\wedge, \vee)$-canonical formulas.

$(\wedge, \vee)$-canonical formulas

Let A be a finite s.i. Heyting algebra, let s be the second largest element of A, and let D be a subset of A^2. For each $a \in A$, introduce a new variable p_a, and set

$$\begin{array}{rcl}\Gamma & = & (p_0 \leftrightarrow \bot) \wedge (p_1 \leftrightarrow \top) \wedge \\ & & \bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\ & & \bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\ & & \bigwedge \{p_{a \to b} \leftrightarrow p_a \to p_b : (a, b) \in D\}\end{array}$$

and

$$\Delta \quad = \quad \bigvee \{p_a \to p_b : a, b \in A \text{ with } a \not\leq b\}.$$

Then define the $(\wedge, \vee)$-canonical formula $\gamma(A, D)$ associated with A and D as

$$\gamma(A, D) = \Gamma \to \Delta.$$

$(\wedge, \vee)$-canonical formulas

If $D = A^2$, then $\gamma(A, D) = \chi(A)$. If $D = \emptyset$, then $\gamma(A, \emptyset) = \gamma(A)$

Theorem. Let A be a finite s.i. Heyting algebra. A Heyting algebra B refutes $\gamma(A)$ iff X_A is an order-preserving image of X_B.

These formulas are counterparts of subframe formulas.

There are continuum many logics axiomatized by such formulas.

Applications of canonical formulas

- In obtaining large classes of logics with the finite model property.
- In proving the Blok-Esakia isomorphism between the lattice of varieties of Heyting algebras and the subvrarites of the Grzegorczyk algebras.
- In showing that the substructural hierarchy of Ciabattoni-Galatos-Terui collapses over superintuitionistic logics.
- In proving that admissibility is decidable over intuitionistic logic and in finding a basis for admissible rules.

Open problems

- Characterize locally finite varieties of Heyting algebras.
- Conjecture: A variety $\mathbf{V}$ of Heyting algebras is locally finite iff $F_{\mathbf{V}}(2)$ is finite.
- Is every variety of Heyting algebras generated by a class of Heyting algebras of the form $\mathrm{Op}(X)$ for some topological space X (Kuznetsov, 1975).

  Heyt is generated by $\mathrm{Op}(\mathbb{R})$ (McKinsey and Tarski, 1946).
- Generalize the theory of $(\wedge, \to)$ and $(\wedge, \vee)$-canonical formulas to other non-classical logics e.g, substractural logics. For modal logics this has been done already.